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INVERSE SOURCE PROBLEMS AND CONTROLLABILITY FOR THE STOKES AND NAVIER-STOKES EQUATIONS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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This thesis is focused on the Navier–Stokes system for incompressible fluids with either Dirichlet or nonlinear Navier–slip boundary conditions. For these systems, we exploit some ideas in the context of the control theory and inverse source problems. The thesis is divided in three parts.

In the first part, we deal with the local null controllability for the Navier–Stokes system with nonlinear Navier–slip conditions, where the internal controls have one vanishing component. The novelty of the boundary conditions and the new estimates with respect to the pressure term, has allowed us to extend previous results on controllability for the Navier–Stokes system. The main ingredients to build our result are the following: a new regularity result for the linearized system around the origin, and a suitable Carleman inequality for the adjoint system associated to the linearized system. Finally, fixed point arguments are used in order to conclude the proof.

In the second part, we deal with an inverse source problem for the N- dimensional Stokes system from local and missing velocity measurements. More precisely, our main result establishes a reconstruction formula for the source $F(x,t) = \sigma(t)f(x)$ from local observations of $N-1$ components of the velocity. We consider that $f(x)$ is an unknown vectorial function, meanwhile $\sigma(t)$ is known. As a consequence, the uniqueness is achieved for $f(x)$ in a suitable Sobolev space. The main tools are the following: connection between null controllability and inverse problems throughout a result on null controllability for the N- dimensional Stokes system with $N-1$ scalar controls, spectral analysis of the Stokes operator and Volterra integral equations. We also implement this result and present several numerical experiments that show the feasibility of the proposed recovering formula.

Finally, the last chapter of the thesis presents a partial result of stability for the Stokes system with Dirichlet boundary conditions and when local and boundary measurements are available, for a source $F(x) = R(x)g(x)$, where $R(x)$ is a known vectorial function and $g(x)$ is unknown. This result involves the Bukhgeim-Klibanov method for solving inverse problems and some topics in degenerate Sobolev spaces.

PROBLEMAS INVERSOS DE FUENTE Y CONTROLABILIDAD PARA LOS SISTEMAS DE STOKES Y NAVIER–STOKES

Esta tesis esta enfocada en el sistema de Navier–Stokes para fluidos incompresibles con condiciones de borde Dirichlet y Navier-slip no lineales. Para estos sistemas, exploramos algunas ideas en el contexto de la teoria de control y problemas inversos de fuente. La tesis esta dividida en tres partes.

En la primera parte, estudiamos la controlabilidad local a cero para el sistema de Navier–Stokes con condiciones Navier-slip no lineales, donde los controles tienen una componente escalar nula. La novedad de las condiciones de borde y las nuevas estimaciones para el termino de presión, nos ha permitido extender anteriores resultados en controlabilidad para el sistema de Navier–Stokes. Las ideas principales para construir nuestro resultado principal son: un nueva resultado de regularidad para el sistema linealizado alrededor de cero, una nueva desigualdad de Carleman para el sistema adjunto asociado al sistema linealizado. Por ultimo, resultados de la teoría de punto fijo son usados para concluir la demostración.

En la segunda parte, abordamos un problema inverso de fuente para el sistema de Stokes en dimension N a partir de mediciones locales y faltantes de la velocidad. Precisamente, nuestro resultado principal establece una formula de reconstrucción para la fuente $F(x,t) = \sigma(t)f(x)$ a partir de observaciones locales de $N-1$ componentes de la velocidad. En la fuente considerada, $f(x)$ es una función vectorial desconocida, mientras que $\sigma(t)$ es una función escalar conocida. Como una consecuencia del resultado anterior, la unicidad de $f(x)$ en cierto espacio de Sobolev es obtenida. Las principales herramientas son: a resultado sobre control a cero para el sistema Stokes con $N-1$ controles escalares, análisis espectral del operador de Stokes y ecuaciones integrales de Volterra. La implementación de nuestros resultados también es presentada junto con varios ejemplos numéricos que muestran la factibilidad de nuestra formula de reconstrucción.

Finalmente, el último capitulo de la tesis presenta un resultado parcial de estabilidad para el sistema de Stokes a partir de mediciones internas y de borde de una componente de velocidad, para una fuente $F(x) = R(x)g(x)$, donde $R(x)$ es una función vectorial conocida y $g(x)$ es desconocida. Este resultado involucra el método de Bukhgeim-Klibanov para resolver problemas inversos y algunos tópicos en espacios de Sobolev degenerados.

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Contents

Chapter 1

General introduction

In many areas of science and technology the mathematical analysis of fluid dynamics plays an important role. For instance, in ship industry, turbomachninery, airplane industry, meteorology, oceanography, medicine, among others. We can begin quickly saying that a fluid consists in a large number of molecules in motion without a precise structure (different to a solid). A first approach to study a fluid might involve writing down the equations of motion for each one of the particles by considering their interactions (for instance, collisions, characterized by the mean free path, but also long-range interactions). In many physical situations, if the mean density of the fluid is not too low, i.e., if the characteristic lengths of the problem are large compared to the mean free path of the particles, then the fluid can be considered as a continuous medium. Thus, the movement of the particles can be considered as a whole and not independently for each particle. Hence, we can define quantities that characterize the system: velocity, density, pressure, and so on.

Additionally, in fluid mechanics there are two classical coordinate system in which the various equations of motion can be written: Lagrangian and Eulerian coordinates. Lagrangian coordinates are associated with a fluid particle (or a fluid volume element) and follow it throughout its evolution. By contrast, Eulerian coordinates are the coordinates of the fixed reference frame associated with the experiment. In other words, the Eulerian description is based on the determination of the velocity of the fluid particle passing through a point x at time t. The Eulerian approach introduced by Euler in the eighteenth century, will be used in this work as the usual framework to study its controllability properties.

1.1 Navier-Stokes equations for an incompressible fluid

Inside of the fluid mechanics we find the Navier-Stokes equations of fluid dynamics, which are a formulation of Newton's laws of motion for a continuous distribution of matter in the fluid state, characterized by an inability to support shear stresses. In this thesis, we present a derivation for the Navier–Stokes system from a viewpoint of the physics elements contained in the equations, although they may be derived systematically from the microscopic description in terms of a Boltzmann equation, with some additional fundamental assumptions. See for instance [Bac67], [Tri12], [Nav23] and[DG95].

The equations of motion

The dependent variables in the so-called Eulerian description of fluid mechanics are the fluid density $\rho(x, t)$, the velocity vector field $u(x, t)$, and the pressure field $p(x, t)$. Here, an $x \in \mathbb{R}^N$ is the spatial coordinate in a N- dimensional space, with $N = 2$ or $N = 3$.

A infinitesimal element of the fluid of volume δV located at position x at time t has mass $\delta m = \rho(x, t)\delta V$ and it is moving with velocity $u(x, t)$ and momentum $\delta m u(x, t)$. The normal force directed into the infinitesimal volume across a face of area $n\delta a$ centered at x, where n represents the unit vector normal to the face, is $-np\delta a$. The pressure is the magnitude of the force per unit area, or normal stress, imposed on elements of the fluid from neighboring elements, see Figure 1.1.

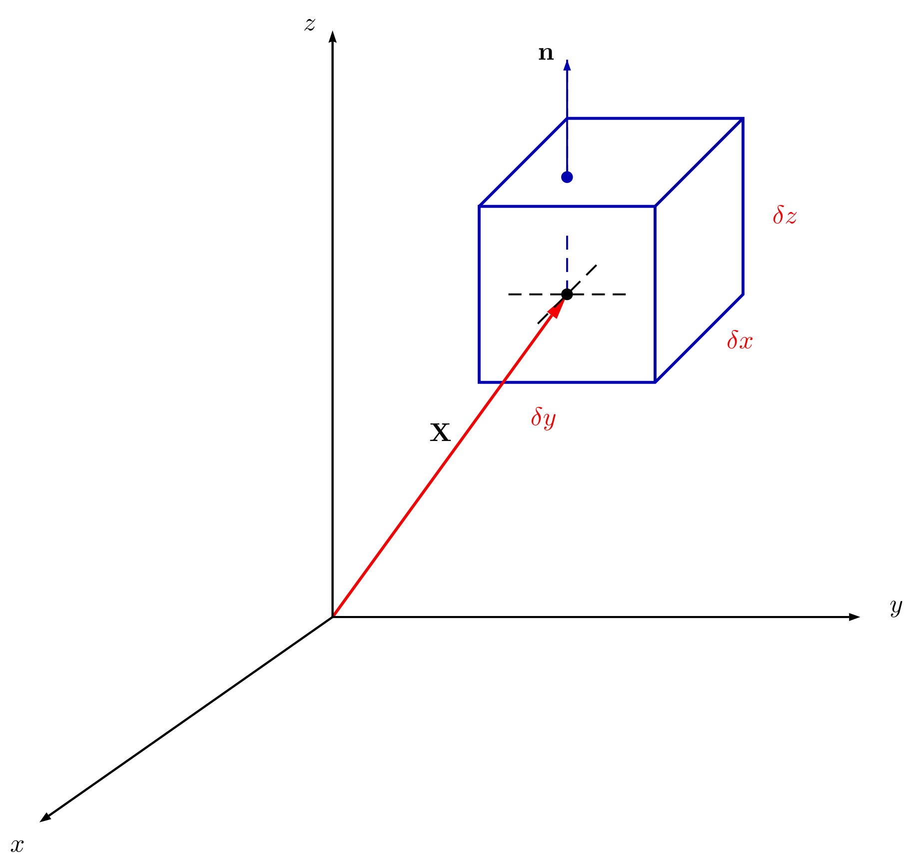

Figure 1.1: A fluid element of volume $\delta V = \delta x \delta y \delta z$ located at position X. The top surface's outward pointing normal n is shown.

On the other hand, the rate of change of a quantity given by the function $f(x,t)$ at a fixed point x in space is simply the partial derivative with respect to time:

$$\frac{\mathrm{d}f(x,t)}{\mathrm{d}t}.$$

However, the rate of change of the same quantity at x, as measured by an observer moving with velocity u is:

$$\frac{\mathrm{d}f(x,t)}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{f(x+u\delta t, t+\delta t) - f(x)}{\delta t} = \frac{\partial f(x,t)}{\partial t} + u\cdot\nabla f(x,t).$$

We refer to this rate of change with respect to an observer moving with the fluid, as the *convective derivative*. Then, we have

$$\frac{\mathrm{d}f(x,t)}{\mathrm{d}t} := \frac{\partial f(x,t)}{\partial t} + u\cdot\nabla f(x,t).$$

Now, we will use the previous definition in the following. Consider the volume δV of an element of mass δm as the system involves. Conservation of mass means that δm does not change for this element. If the element compress or expands then the volume and density will change, but the mass is fixed:

$$\frac{\mathrm{d}\delta m}{\mathrm{d}t} = 0. \tag{1.1}$$

The rate of change of the volume occupied by δm is given by (see [Tri12]):

$$\frac{\mathrm{d}\delta V}{\mathrm{d}t} = (\nabla\cdot u)\delta V. \tag{1.2}$$

Hence the divergence $\nabla\cdot u$ of the velocity vector field is the local rate of change of the volume of elements of mass. In terms of the density ρ this corresponds to:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta m}{\delta V} = -\frac{\delta m}{(\delta V)^2}\frac{\mathrm{d}\delta V}{\mathrm{d}t} = -\rho\nabla\cdot u. \tag{1.3}$$

Then, using the previous definition of convection derivative, we see that conservation of mass manifests itself as the continuous equation:

$$0 = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u). \tag{1.4}$$

From Newton's second Law of motion, which states that the rate of change of momentum equals the net applied force, can be applied to each element of mass in the fluid. Thus, in the absence of any externally applied forces, the net force δF acting on each element of mass is due to the pressure field. Then, the component of force in the x direction is (see Figure 1.2):

$$\delta F_1 = p(x - \hat{\imath}\delta x/2, t)\delta y\delta z - p(x+\hat{\imath}\delta x/2, t)\delta y\delta z = -\frac{\delta p}{\delta x}\delta V. \tag{1.5}$$

Therefore, Newton's second law for the element of mass δm at position x and time t is

$$\frac{\mathrm{d}(\delta m u(x,t))}{\mathrm{d}t} = \delta F = -\delta V\nabla p. \tag{1.6}$$

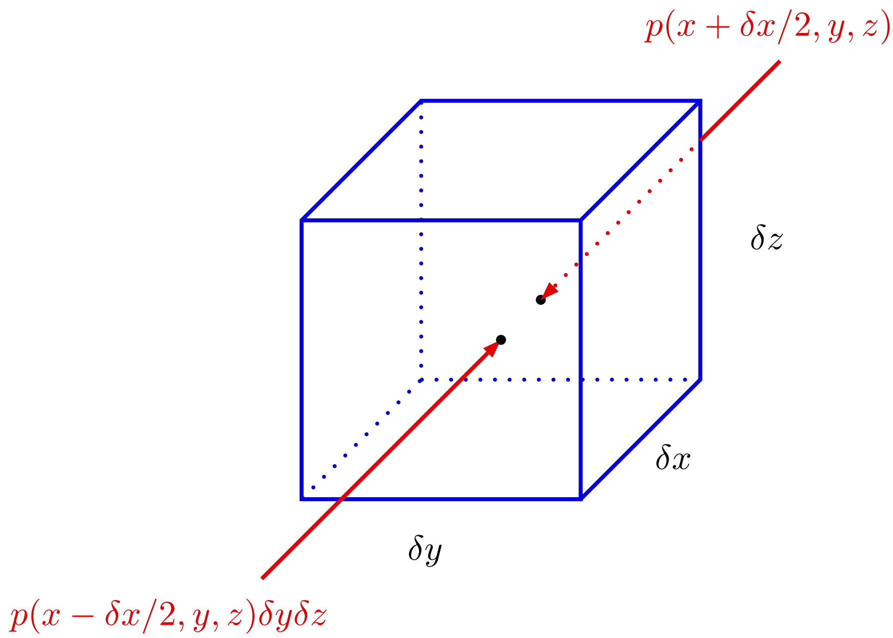

Figure 1.2: The pressure force acting on the front and rear of a fluid element

Recalling the equation of conservation of mass (1.1) and dividing through by δm we deduce the equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p, \tag{1.7}$$

called the *Euler's equations*. Therefore, by combining the Euler's equations and the continuity equation (1.4), we obtain $N + 1$ evolution equations for the $N + 2$ dependent variables (N components of the velocity u, the density ρ and the pressure p). What remains is to provide a relationship between the density and pressure. Typically this given in the form of a thermodynamic equation of state. For example, in an ideal gas at constant temperature, $p \approx \rho$.

A significant simplification is achieved by considering fluids which are effectively incompressible, but, does this mean?. Physically, this condition is applied to problems where all the relevant velocities are much smaller than the speed of sound in the fluid. The continuity equation (1.4) then implies that the derivative of the density vanishes, so the density of each fluid element never changes from its initial value, so that

$$\rho(x, 0) = cte \Rightarrow \rho(x, t) = cte.$$

In synthesis, the flow of a fluid is said to be incompressible if one of the following equivalent properties is satisfied (see [BF12]):

i) The volume of any fluid element is constant along the time.

ii) The velocity field u is divergence-free (it is also said to be *solenoidal*):

$$\nabla \cdot u = 0.$$

iii) The density ρ is constant along the trajectories associated with the velocity field u.

Then, Euler's equations for an incompressible homogeneous fluid are:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0 \tag{1.8}$$

and

$$\nabla \cdot u = 0, \tag{1.9}$$

where the density is now a parameter and moreover we have $N+1$ equations for the $N+1$ unknowns variables. Observe that a flow can be incompressible even if the density is not constant. It is only required that the density of a particle of fluid remain constant during the evolution. As an example of a non homogeneous incompressible flow we can consider water in the ocean, whose density depends on the salinity but which is nevertheless incompressible.

In order to derive the Navier-Stokes equations, it is necessary to consider the viscosity in the fluid. Viscosity is a measure of the diffusion of momentum due to the microscopic molecular nature of real fluids, and its effect is to produce a resistance to shearing motions. As such, it is a frictional force with its origins in the microscopic interactions between the atoms or the molecules making up the fluid. Its net effect is to dissipate in an organized way, macroscopic forms of energy - the kinetic energy in the flow field - and convert it to the disorganized, microscopic form of energy, heat (see [Tri12] for more details). Shearing forces in continuum mechanical systems are described by the stress tensor. The tensional nature of these forces results from the fact that there are two directions associated with each such force, the direction of the force itself and the orientation of the area across which the force acts.

Consider a rectangle shaped portion of fluid, centered at the point (x, y, z) with side lengths $(\delta x, \delta y, \delta z)$, as Figure 1.3. The component σ_{ij} of the stress tensor σ is the force per unit area in the jth direction acting across an area element whose normal is in the ith direction. Forces in the direction of the normal to an area element are associated with the pressure, while those that acts in the plane of the element are associated with shear stresses. Newton's third law implies that forces of equal magnitude and opposite direction act on the sides due to the matter on the sides. Adding these forces, the net force on the fluid element in the jth direction is

$$\begin{aligned} \delta F_j \quad = \quad & \sigma_{1j}(x + \delta x/2, y, z)\delta y \delta z - \sigma_{1j}(x - \delta x/2, y, z)\delta y \delta z \\ & +\sigma_{2j}(x, y + \delta y/2, z)\delta x \delta z - \sigma_{2j}(x, y - \delta y/2, z)\delta x \delta z \\ & +\sigma_{3j}(x, y, z + \delta z/2)\delta x \delta y - \sigma_{3j}(x, y, z - \delta z/2, z)\delta x \delta y. \end{aligned}$$

Hence the force per unit volume acting at a point in the fluid due to stress within the fluid is the divergence of the stress tensor, i.e.,

$$\frac{\delta F}{\delta V} = \nabla \cdot \sigma.$$

When the torque acts on the volume element due to the stress tensor σ, the z component of torque is:

$$N_3 = k \cdot \sum_{\text{faces}} r \times \delta F = (\sigma_{12} - \sigma_{21})\delta x \delta y \delta z$$

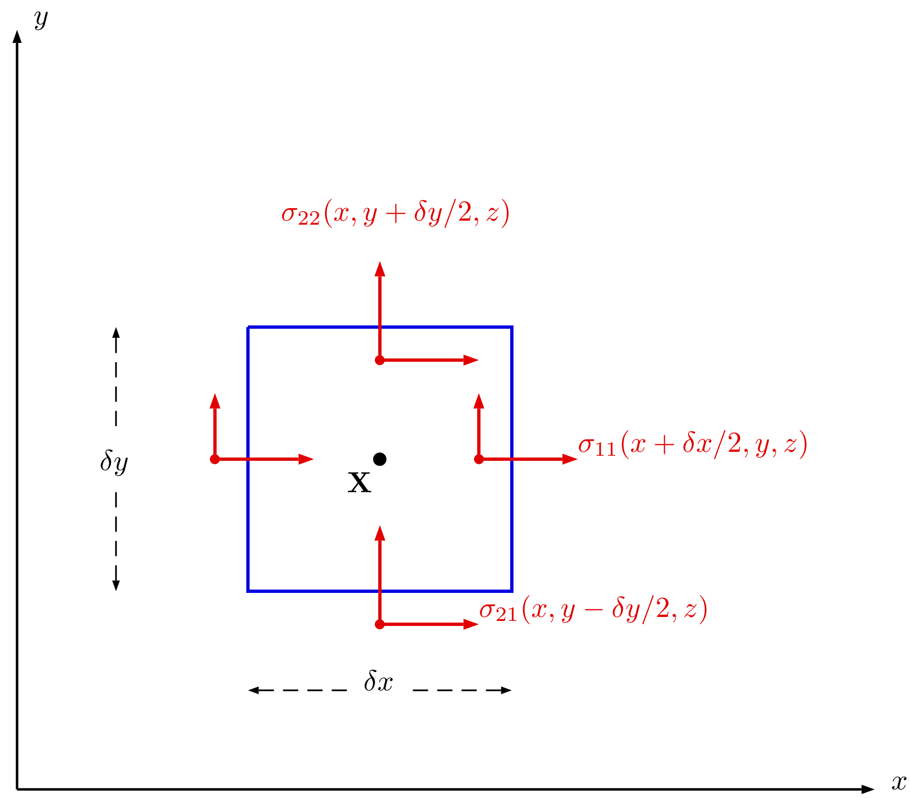

Figure 1.3: Several components of the stress tensor acting on a fluid element located at X. The force acts on the sides of the faces of the element as indicated by the positions of the vectors in red. For example, the horizontal force acting on the element due to the stress at the bottom face is $-\sigma_{21}(x, y - \delta y/2, z)\delta x \delta z$.

and the z component of the inertia tensor is

$$I_{33} = \frac{1}{24}(\delta x^2 + \delta y^2)\rho \delta x \delta y \delta z.$$

Then, typical associated angular accelerations are

$$\frac{N_3}{I_{33}} \approx \frac{\sigma_{12} - \sigma_{21}}{\rho} \frac{1}{\delta x^2 + \delta y^2}.$$

The necessity if a symmetric stress tensor is then apparent in order to realize a consistent continuum limit as $\delta x \to 0$, $\delta y \to 0$ and $\delta z \to 0$.
The stress tensor can be represented into portions due to the pressure p and the symmetric stress tensor $T_{\mathrm{i}j}$, that is

$$\sigma_{\mathrm{ij}} = -\delta_{\mathrm{i}j} p + T_{\mathrm{i}j}, \tag{1.10}$$

where $\delta_{\mathrm{i}j}$ is the Kronecker Delta function. Thus, the general form of the equation of motion for the velocity vector field u, referring to (1.5)-(1.7) is

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\nabla \cdot T. \tag{1.11}$$

The rate of strain tensor may be defined as that controlling the evolution of the relative positions of points in a fluid element. Let δx denote infinitesimal displacement of two points in the fluid, one at x and the other at $x+\delta x$. The rate of change of $|\delta x|^2$ corresponds to:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\delta x|^2 = 2\delta x \cdot [u(x+\delta x) - u(x)] = \delta x_{\mathrm{i}} \Big(\frac{\partial u_{\mathrm{i}}}{\partial x_j} + \frac{\partial u_j}{\partial x_{\mathrm{i}}} \Big) \delta x_j = \delta x \cdot D \cdot \delta x,$$

where D represents the symmetric rate of the strain tensor (or symmetrized gradient). On the other hand, the relationship between D and T is (see for instance [BF12]):

$$T = \alpha D + \beta Tr(D) I, \tag{1.12}$$

where I is the unit tensor and the constant α, β are material parameters. The components of the viscous force per unit volume are then

$$(\nabla \cdot T)_{\mathrm{i}} = \alpha \Delta u_{\mathrm{i}} + (2\beta + \alpha) \frac{\partial}{\partial x_{\mathrm{i}}} \nabla \cdot u.$$

From the incompressible condition (1.9) and previous identity, we obtain the *incompressible Navier-Stokes equations*

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho} \nabla p = \nu \Delta u \tag{1.13}$$

$$\nabla \cdot u = 0, \tag{1.14}$$

where ν is the kinematic viscosity. Compared to the incompressible Euler equations, the net effect of the linear coupling between stress and rate of strain is to introduce the “ diffusion” term at the right-hand side of (1.13). The diffusion of momentum between neighboring elements of the fluid is indeed a new ingredient in the incompressible Navier-Stokes equations, but there is also the matter of initial and boundary conditions that we will see in the following paragraph.

Impermeable boundaries and initial condition

If the fluid is confined to a fixed region of space Ω bounded by $\partial\Omega$, the fluid cannot cross the rigid boundaries. Thus, think that perhaps the simplest type of boundary is an impermeable wall is appropriate, such as the side of a wake-tank or the hull of a ship. If the boundary $\partial\Omega$ is stationary, then the appropriate boundary condition for an fluid is

$$u \cdot n = 0 \quad \text{on } \partial\Omega,$$

where n represent the unit outward normal vector to the boundary. This ‘no-flow’ condition states that the fluid does not flow through the boundary. An fluid can ‘slide’ over an impermeable boundary, and the tangential velocity is, in general, nonzero. However, observing that the Navier-Stokes equation contain second-order spatial derivatives, they require additional boundary conditions. The most usual used condition is a Dirichlet boundary condition for the velocity, represented by

$$u = u_b \quad \text{on } \partial\Omega.$$

When $u_b = 0$, it is called an homogeneous Dirichlet boundary condition or a *no-slip* condition (this means that the fluid ‘sticks’ to the boundary).

On the other side, there are numerous researchers which have cast doubts on the universality of the no-slip boundary conditions, showing that under certain circumstances fluid slip might occur at the solid boundary (see for instance [Bac67], [LBS07]). In presence of slip conditions, C.L. Navier proposed in 1823 the Navier-slip boundary conditions by establishing that the component of velocity tangential to the surface should be proportional to the tangential component of the rate of stress at the surface, i.e.,

$$u \cdot n = 0, \quad (\sigma(u,p) \cdot n)_{tg} = k(u)_{tg} \quad \text{on } \partial\Omega,$$

where σ was introduced in the previous section and tg stands for the tangential component of the corresponding vector field, i.e. (see [Nav23]):

$$w_{tg} = w - (w \cdot n)n.$$

In most of the situations, the Navier-slip boundary condition can be reduced to the no-slip boundary conditions due to extremely small slip length. However, in some cases as in the driven cavity flow problem or some turbulence problems, it has been shown that the Navier-slip boundary condition is valid and removes un-physical singularities (see for instance [Bac67], [LBS07] and [Pan06]).

Finally, as the Navier-Stokes equations are an unsteady model, it is required to impose initial conditions in order to define the evolution of the system, evidently in a suitable Banach space. It has no mathematical meaning to impose an initial value for the pressure because this unknown has the role of the Lagrange multiplier associated with the incompressible condition and thus, is defined in some indirect way, see for more details the books [Tem01] and [BF12].

In this thesis we consider homogeneous Dirichlet boundary conditions for the inverse source problems of the Stokes system in Chapter 3 and Chapter 4, and nonlinear Navier-slip boundary conditions for the control problem presented in Chapter 2.

On the existence, uniqueness and regularity of solutions

There is an extensive literature on this subject since the pioneer work of J. Leray in [Ler33]-[Ler34], where he introduced many fundamental ideas. In [Ler34] he constructed a global (in time) weak solution and a local strong solution of the initial value problem when $\Omega = \mathbb{R}^3$. On the other side, H. Hopf proved the existence of a global weak solution of the initial-boundary value problem. Such solutions are called *Leray-Hopf solutions*. When the dimensional space is $\mathbb{R}^2$ the Leray-Hopf solutions are unique and regular, see the works [Lio69], [LP59], [LS69], [Ser63], [Tem01]. However, for $N = 3$ the uniqueness and regularity of Leray-Hopf solutions are still important open problems.

On the other hand, although the energy estimate for solutions is fundamental to prove that there is a global weak solution, meanwhile, if we discuss the existence of a unique local strong solution, the semigroup method introduced in [FK64], [KF62] is more powerful that the energy estimate, so each method has advantages and disadvantages. In fact, when $N = 2$, the energy estimate is strong sufficient to prove the global existence of smooth solutions, however, when $N = 3$ the energy method has been not capable to provide such a result. If $N = 3$, it is possible to estimate the size of possible singular set of Leray-Hopf solutions, using the energy estimate. The reader interested in this topics can review [CKN82], [Sch78] and [Lio96] for more details.

1.2 Some aspects of the controllability in PDE's

In general aspects, the control problem consists in given two states of the system determine whether is possible to drive the establish system from the first state to the given second state by means of an applied control to the system.
We consider an abstract linear dynamic system

$$\begin{aligned} &\frac{\partial y}{\partial t} + \mathcal{A}(y) = \mathcal{B}h, \\ &y(\cdot, 0) = y_0 \in \mathcal{H}, \end{aligned} \tag{1.15}$$

where y is the *variable state* in the state space $\mathcal{H}$. The dynamic of the system depends of the parameter h, called *control function*, thanks to which we can act on the evolution of the state. The question that we ask is: is it possible for a given time $T > 0$ and two states of the system y_0, y_1 to find a control h such that the solution y of (1.15) starting $y(0) = y_0$ satisfies $y(T) = y_1$?. The properties of controllability for the system (1.15) can be different depending on the nature of the problem. In general terms we can distinguish the control in ordinary differential equations (or finite-dimensional controls), the control in partial differential equations (or infinite-dimensional controls), the control in linear and nonlinear equations. In this section we briefly describe some classical problems of the controllability for infinite-dimensional systems modeled by partial differential equations.
We assume here that $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is an operator with domain $\mathcal{D}(A) \subset \mathcal{H}$. By $\mathcal{Y}$ we denote the control space and $\mathcal{B} \in \mathcal{L}(\mathcal{Y}, \mathcal{H})$.

There are many physical problems associated to the abstract framework (1.15), in particular the oscillate system (wave equation) and dissipative system (heat equation), for which we recall some results below.
In the following, we assume that the Cauchy problem associated to (1.15) is well-posed (without considering the control problem), that means, assume that the operator $\mathcal{A}$ generates on $\mathcal{H}$ a strongly continuous semigroup denoted by $S(t) = \mathrm{e}^{tA}$, with $S \in C^0(\mathbb{R}_+; \mathcal{L}(\mathcal{H}))$ (see for instance [Paz12]). In contrast to the case of linear finite-dimensional control systems, see the book [Son13], in the infinite-dimensional case many types of controllability are possible. We define here three types of controllability.

Definition 1.1 *Let $T > 0$. The control system* (1.15) *is exactly controllable in time T if, for every $y_0 \in \mathcal{H}$ and for every $y_1 \in \mathcal{H}$, there exists $h \in L^2(0, T; \mathcal{Y})$ such that the solution y of the Cauchy problem* (1.15) *satisfies $y(T) = y_1$.*

Definition 1.2 *Let $T > 0$. The control system* (1.15) *is null controllable in time T if, for every $y_0 \in \mathcal{H}$ and for every $\tilde{y}_0 \in \mathcal{H}$, there exists $h \in L^2(0,T;\mathcal{Y})$ such that the solution of the Cauchy problem* (1.15) *satisfies $y(T) = S(T)\tilde{y}_0$.*

Let us point out that, by linearity, we get an equivalent definition of null controllability in time T if, in the previous definition one assumes that $\tilde{y}_0 = 0$. This explains the usual terminology null controllability.

Definition 1.3 *Let $T > 0$. The control system* (1.15) *is approximately controllable in time T if, for every $y_0 \in \mathcal{H}$, for every $y_1 \in \mathcal{H}$, and for every $\varepsilon > 0$, there exists $h \in L^2(0,T;\mathcal{Y})$ such that the solution y of the Cauchy problem* (1.15) *satisfies $\|y(T) - y_1\|_{\mathcal{H}} \leq \varepsilon$.*

Clearly exact controllability implies null and approximate controllability. However, when S is a strongly continuos group of linear operator the converse is true, but in general aspects, the converse is false (see [Cor07], section 2.3.2).

Generally the controllability of a system is difficult to prove it directly, so it is convenient to introduce an alternative method, the principal is called observability.
Let us introduce the system

$$\begin{cases} \dfrac{\partial w}{\partial t} = \mathcal{A}^* w & \text{in } (0,T), \\ c(t) = \mathcal{B}^* w(t) & \text{in } (0,T), \\ w(T) = w_T \in \mathcal{H}, \end{cases} \tag{1.16}$$

where $\mathcal{A}^*, \mathcal{B}$ are the adjoint operators of $\mathcal{A}$ and $\mathcal{B}$ respectively, and we assume that the problem is well-posed backwards in time. In fact, the adjoint semigruop $S^*(t) = \mathrm{e}^{(T-t)\mathcal{A}^*}$ is generated by $\mathcal{A}^*$ and the solution for the previous system can be written as $w(t) = S^*(t)w_T$.
The observability problem is the following: is it possible by observing only the quantity $c(t)$, to know the energy of the system (1.16) at final time $t = 0$, that is say $\|w(0)\|^2_{\mathcal{H}}$?.

Definition 1.4 *The system* (1.16) *is observable in time $T > 0$ if there exists a constant $C > 0$ such that for every $w_T \in \mathcal{H}$, the solution of* (1.16) *satisfies*

$$\|\mathrm{e}^{T\mathcal{A}^*} w_T\|^2_{\mathcal{H}} = \|w(0)\|^2_{\mathcal{H}} \leq C \int_0^T \|B^* w(t)\|^2_{\mathcal{H}} \mathrm{d}t. \tag{1.17}$$

This notion of observability is useful in many concrete situations when we wish to know the state of the system from partial measurements, this is the case for example in meteorology, images and more generality in the domain of inverse problems.
Another interest of observability resides in its connection with the controllability. We must make the following assumptions of retrograde uniqueness (which verifies every linear system in this thesis): every solution of (1.16) that satisfies $w(0) = 0$ is identically zero.
It is known that J.-L. Lions in [Lio88] (among others authors) proved that the system (1.16) is observable in time T if and only if the system (1.15) is controllable to zero in time T. The proof of this result is based on a mathematical method called Hilbert Uniqueness method

(HUM).

Wave equation

There are many examples of wave equations in the physical sciences, characterized by oscillating solutions that propagate through space and time while, in lossless media, conserving the energy. Examples include the scalar wave equation (pressure waves in a gas), Maxwell's equations (electromagnetism), Schrodinger's equation (quantum mechanics), elastic vibrations, and so on.
It is important to say that the wave equation is the most relevant hyperbolic partial differential equation, where the main properties of hyperbolic equations such as time-reversibility and the lack of regularizing effects, have some important consequences in control problems (see for instance [Pue11]).

There is a huge literature on the controllability of linear wave equations for any space dimension. One the best results on this subject has been obtained in [BLR88] and [BLR92] for the system:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = h1_{\omega\times(0,T)} & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y = y_0, \quad \dfrac{\partial y}{\partial t} = y_1 & \text{in} \quad \Omega \times \{t=0\}, \end{cases} \tag{1.18}$$

where h represents the control function that acts on the open subset ω of Ω in time interval $(0,T)$. In these papers the authors proved that, in the class of C^∞ domains, the observability inequality associated to the previous system (for the null controllability) holds if and only if (ω, T) satisfy the following *geometric control condition* (GCC) in Ω: *every ray of geometric optics that propagates in Ω and reflected on its boundary $\partial\Omega$ enters ω in time less than T.* For instance, for a square domain Ω, observability (controllability) fails if the control is supported on a set which is strictly smaller than two adjacent sides.

There are of course many other references which deal with the controllability of hyperbolic equations. See for instance the paper [GL] by Robert Gulliver and Walter Littman, the books [FI96b] by A. Fursilov and O. Imanuvilov, [Lio91] by J.-L. Lions and [Kom94] by V. Komornik, where one can find different results and useful references.

Heat equation

The heat equation governs heat diffusion, as well as other diffusive process, for instance, the temperature distribution and evolution in a body occupying the region Ω, particle diffusion and so on. Some aspects respect to the control problems are described below. To get an idea, let us consider the case of the linear heat equation with Dirichlet homogeneous boundary conditions and distributed controls:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = h 1_{\omega \times (0,T)} & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(\cdot, 0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{1.19}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^2, $\omega \subset \Omega$ is an open set on which acts the control h (h is a localized source of heat) and y_0 is the initial state, for instante, y_0 in $L^2(\Omega)$. The system (1.19) are characterized by *nonreversibility*, the *dissipativity* of the solutions, that is, the fact that energy is lost along the trajectories, and the *regularizing effect*. Taking into account the regularizing effect, it is not possible to drive the solutions of (1.19) exactly for every final state in a suitable Sobolev space, except in the trivial case when $\omega = \Omega$, which is not interesting. In this sense, the notion of null controllability is not relevant for parabolic equations. Thus, the good notion of controllability is not to go from a given state to another state in a fixed time, but to go from a given state to a given trajectory (notion equivalent to the null controllability introduced in definition 1.2).

One can find that the controllability problems for parabolic equations has been analyzed in several papers, among them, [LR95] where the author proved null controllability for system (1.19) using in the spectral properties of the Laplacian operator in order to construct a control h. Also in [FI96b] the null controllability for the system (1.19) is obtained, but through an observability inequality for the adjoint system, where the main tools are Carleman inequalities. For another parabolic equations (linear and nonlinear) and its study in controllability, the reader can see [FPZ95], [Bar00], [FCGBGP06], [Lio91] and [FI96b] for more details.

1.3 Controllability for the Navier–Stokes equations

In this section we mention the different problems that exist in controllability for the Navier-Stokes equations. The first idea corresponds to the global controllability results for incompressible fluids modeling for this equations, which are based in the *return method*. Briefly, this method goes as follows: Can find a trajectory of the nonlinear control system such that

a) It stars and ends at the equilibrium.
b) The linearized control system around this trajectory is controllable.

Thus, thanks to the implicit function theorem one can go from every state close to the equilibrium to every other state close to the equilibrium.

Now, in order to define some aspects of the controllability for the Navier–Stokes equations, we introduce some notation. Let $N = 2$ or $N = 3$ and let Ω be a bounded nonempty connected open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $\Gamma_0 \subset \partial\Omega$ and ω_0 be an open subset of Ω where the control acts.

Definition 1.5 *A trajectory of the Navier-Stokes control system on the time interval* $[0,T]$ *is a map* $\overline{y} : [0,T] \times \overline{\Omega} \to \mathbb{R}^N$ *such that, for some function* $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$*, the pair* $(\overline{y}, p)$ *satisfies the system* (1.13) *in* $[0,T] \times \overline{\Omega} \backslash \omega_0$ *with divergence free condition* (1.14) *in* $[0,T] \times \overline{\Omega}$*, and* $\overline{y}(\cdot, t)$ *safisfies the boundary conditions on* $\partial\Omega \backslash \Gamma_0$*.*

The Jacques-Louis Lions problem of approximate controllability is the following:
Problem. Starting with the initial data y_0 for the velocity field, we ask whether there are trajectories of the Navier-Stokes system which, at a fixed time T, are arbitrarily close to the given velocity field y_1. In other words, for $T > 0$, consider $y_0, y_1 : \overline{\Omega} \to \mathbb{R}^N$ satisfying (1.14) and boundary conditions on $\partial\Omega \backslash \Gamma_0$, the question is, does there exists a trajectory $\overline{y}$ of the Navier-Stokes control system such that

$$\overline{y}(\cdot, 0) = y_0 \quad \text{in } \overline{\Omega}, \tag{1.20}$$

and, for an appropriate topology (see [Lio91]),

$$\overline{y}(\cdot, T) \text{ is close to } y_1 \text{ in } \overline{\Omega}?. \tag{1.21}$$

If the previous problem has a solution, we say that the system is approximately controllable. If we change the condition (1.21) by

$$\overline{y}(\cdot, T) = y_1 \quad \text{in } \overline{\Omega}, \tag{1.22}$$

it is possible to prove as consequence of the smoothing effect of the Navier-Stokes system that the problem does not admit solution for arbitrary y_1. Thus, we replace (1.22) by another condition in order to recover a natural definition of controllability for the Navier-Stokes system. A better definition for controllability, which was presented in [FI95] and [CF96] is passing from a given state y_0 to a given trajectory $\hat{y}_1$. Then, the control problem for the Navier-Stokes system with Stokes or Navier-slip conditions can be written as follow.

Problem. Let $T > 0$. Let $\hat{y}_1$ be a trajectory for the Navier-Stokes system on $[0,T]$. Let $y_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ satisfy the divergence free condition (1.14) and the boundary conditions used. Does there exist a trajectory $\overline{y}$ of the Navier-Stokes system on $[0,T]$ such that

$$\overline{y}(x, 0) = y_0(x) \quad \text{and} \quad \overline{y}(x, T) = \hat{y}_1(x), \quad \forall x \in \overline{\Omega}? \tag{1.23}$$

Related to this problem, one knows two types of results: local results and global results.

The local results do not rely on the return method and instead are related with observability inequalities for the heat equation. The main difficulty here is to estimate the pressure term. The definition of local controllability along trajectories for the Navier-Stokes system is the following:

Definition 1.6 *The Navier-Stokes system is locally controllable along the trajectory* $\hat{y}_1$ *on* $[0,T]$ *of the Navier-Stokes control system if there exists* $\varepsilon > 0$ *such that, for every* $y_0 \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ *satisfying* (1.14)*, boundary conditions and*

$$\|y_0 - \hat{y}_1(\cdot, 0)\|_{H^1(\Omega)^N} < \varepsilon,$$

there exists a trajectory $\overline{y}$ *of the Navier-Stokes system on* $[0, T]$ *satisfying* (1.23).

We mention that the local controllability for the Navier-Stokes system has been studied for many mathematicians. The main works are:

a) The papers [FI94] and [FI96a] where the authors treated the case $N = 2$ and linear Navier-slip boundary conditions.

b) In [Fur95] the author treated the case where $\Gamma_0 = \partial\Omega, N = 3$ and Dirichlet boundary conditions.

c) In [Ima01] the author proved the case of the homogeneous Dirichlet boundary conditions.

d) In [FCGIP04] the authors weakened some regularity assumptions.

e) In [Gue06] the author proved the case of the Navier-slip boundary conditions.

f) The work presented in [CG13], where the authors proved the local null controllability for the Navier-Stokes system with one vanishing component in the control.

The global controllability results are usually much more complicated than getting local controllability results. We find in [Cor96] a proof based in the return method. Let us recall that it consists of looking for a trajectory of the Navier-Stokes system $\overline{y}$ such that

$$\overline{y}(\cdot, 0) = \overline{y}(\cdot, T) = 0 \quad \text{in } \overline{\Omega},$$

and such that the linearized system around the trajectory $\overline{y}$ has a controllability in a good sense.

Finally, in [FI94] and [FI96a] the authors proved that the lineriazed system around the trajectory $\overline{y}$ with Navier-slip boundary conditions is controllable. On the other hand, in [Lio71] is proved the approximate controllability, meanwhile, in [AS05], [Shi06] the authors have obtained global approximate controllability results for the Navier-Stokes equations (also for the Euler equations) when the controls are on some low modes and Ω is a torus.

There are other papers that deal with the interaction fluid with other materials. For instance:

a) The papers [OP99], [Luk72] and [LZ96] on the controllability of an incompressible fluid interacting with an elastic structure.

b) In [DFC05] the authors treated with the controllability of one-dimensional nonlinear system which models the interactions of a fluid and a particle.

c) The controllability of a model linearized and simplified 1-D model for fluidstructure interaction is studied in [ZZ03].

1.4 Inverse problems in PDE's

The origin of the term *inverse problem* (around 1960s) is simple and mirrors what is called the forward (or direct) problem. In simple terms, the direct problem is the situation: given the questions, find the answer, whereas the inverse problem is given the answer, find the question. Thus, an inverse problem consists in to determine a cause from its effect. However, in some cases, there is no hope of ever being able to solve the direct problem in full generality. Many applications of inverse problems can be found in the physical and mechanics sciences: biomedical engineering (ultrasound, X-ray), acoustics, radioastronomy, imaging, meteorology, oceanography, oil engineering, seismology, so on. It is probably fair to say that the majority of real world problems are inverse problems.

The French mathematician Jacques Hadamard introduced in 1923 the term *well- posed* for a mathematical problem where: the solution always exists (existence), the solution is unique (uniqueness) and, small changes in the initial conditions leads small changes in the solution (the solution depends continuously on the data). The opposite case of a well posed problem is called *ill-posed*, this means that, a solution may not exist, there may be more than one solution, small changes in the initial conditions leads to big changes in the solution. The inverse problems tend to be ill-posed.
If the data from measurements can in theory create a space of either finite or infinite dimensions, in practice the data are always finite and discrete. When the number of parameters in a model is smaller than the number of data points from the measurements, the problem is called *overdetermined*. In that case, it may be possible to add a criterion that diminishes or eliminates the effect of aberrant data. On the other hand, if the problem consists in determining continuous parameters that are thus sampled from a very large number of values, and if the number of results from the experiments is insufficient, the problem is called *underdetermined*. It is then necessary to use a priori information to achieve a reduced number of possible solutions, or, in the best case, only one. Since for an underdetermined problem there are often several possible solutions, it is necessary to specify the confidence level that one can give to each solution. For these problems, the data can also be affected by a likelihood coefficient (or probabilistically weighted), if this is the case, a *Bayesian approach* can be used for the problem. The following scheme allows to clarify some previous ideas even further.

Definition 1.7 *Let* $(V_1, \|\cdot\|_{V_1})$ *and* $(V_2, \|\cdot\|_{V_2})$ *be two normed vector spaces and* $F : V_1 \to V_2$ *be a given mapping. The direct problem is to determine* $y = F(x)$*, when* $x \in V_1$ *is given. The inverse problem is to determine such* $x \in V_1$ *that* $y = F(x)$ *when an arbitrary* $y \in V_2$ *is given. The mapping* F *is called the direct theory.*

The previous abstract inverse problem is well-posed whether there exists a solution, the solution has to be unique and the inverse mapping $F^{-1} : V_2 \to V_1$ (if there exists) has to be continuous. More precisely:

- *Existence.* For every $y \in V_2$ there has to be $x \in V_1$ such that $y = F(x)$. In other words, the direct problem needs to be a surjection. Thus, arise the problem to characterize those $y \in V_2$ that correspond to unknown $x \in V_1$.

- *Uniqueness.* If $x_1, x_2 \in V_1$ are two solutions satisfying $F(x_1) = F(x_2)$ in V_2, then $x_1 = x_2$ has to hold. That means, the direct theory needs to be an injection. Therefore, arise the question whether is there enough data to determine the solution uniquely?. This problem is called *identificability*.

- *Continuous dependency on the data.* When F is injective and surjective, then the inverse mapping $F^{-1} : V_2 \to V_1$ has to be continuous. Now the problem is, how small changes in the data disturb the corresponding mathematical solutions?. This is called a *stability problem*.

However, there are two additional problems: how x is obtained from the given y in $F(V_1)$, and of course, an approximative method for recovering the unknown available data. These problems correspond to the theoretical and numerical *reconstruction*.
In finite dimensional linear inverse problems the direct mappins F can be represented with the help of a matrix M. Here, the inverse problem is well-posed if
- For every $y \in V_2$ the equation $y = M(x)$ has a solution $x \in V_1$.
- The equation $Mx = 0$ has only the trivial solution.
On the other hand, the inverse problem is ill-posed if at least one of the following claims holds:
- For some $y \in V_2$ the equation $y = Mx$ does not have a solution $x \in V_1$.
- There exists $x \in V_1$ that satisfies $Mx = 0$ and $x \neq 0$.
If the data contains too much disturbances, the solution of a well-posed problem can be far from the true solution. A well-posed problem which is highly "ill-conditioned" can resemble an ill-posed problem where the solution does not depend continuously on the data.
Finally, it is clear that in infinite dimensional this questions are more complicated than in finite dimensional, but many real phenomenon are describe in this context.

To start out with a concrete description on an inverse problem, we comment the classical inverse problem of gravimetry. The simplest equation that represents the strength of a gravitational field u in $\mathbb{R}^3$ is given by

$$-\Delta u = f, \tag{1.24}$$

where f is the mass distribution that generates the measurements of the gravitational force ∇u, and which is considered outside a bounded domain Ω as zero. Here, Ω is a ball or a body close to a ball (earth). The *direct problem* in gravimetry is to find u given f. This is a well-posed problem in the Hadamard sense: its solution exists for any integrable function f, and even for any distribution that is zero outside Ω; it is unique and stable with respect to standard functional spaces. The solution is given by

$$u(x) = \int_\Omega k(x-s) f(s) \mathrm{d}s,$$

where k is a specific kernel. On the other hand, the *inverse problem of gravimetry* is to find f given ∇u on Γ, where Γ is a part of the boundary $\partial\Omega$ (gravitational force on the boundary). Physically, this inverse problem is fundamental in recovering the density of the earth from boundary measurements of the gravitational field. Another interesting application is in gravitational navigation: one can measure the gravitational field (from satellites) with quite high precision, then possibly find the function f that produces this field, and use this result to navigate aircrafts. However, there is a strong non uniqueness of f for a given gravitational potential u outside Ω, and therefore the uniqueness of the inverse source problem is restricted to a special type of f, for instance harmonic functions, functions dependent on one variable, characteristic functions with unknown domains inside Ω. Thus, the inverse problem of gravimetry is ill-posed, which creates mathematical and numerical difficulties (convergence of iterative algorithms is very slow and therefore numerical errors accumulate do not allow a good resolution). The Victor Isakov books [Isa06] and [Isa90] contains partial results for the inverse problem presented above and other classical inverse problems. The work of Addellatif and Doung [EBD98] deal with the problem of *identification of source* from boundary measurements for the system (1.24). Furthermore, we highlight Theorem 4.1.6, presented in [Isa06], which will have connection with our main result in Chapter 4. Theorem 4.1.6 is referent to the following linear inverse source problem: Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let us consider the Dirichlet problem

$$\begin{cases} Au = f & \text{in} \quad \Omega, \\ u = g_0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.25}$$

where $A = \partial_j(a\partial_j) + c$, for every $j = 1, \ldots, N$. Let $\mathcal{L}$ be the differential operator $\partial_j(\alpha_j\partial_j) + \beta$.

Theorem 1.8 *Let us assume that one of the three conditions is satisfied:*

$$\begin{gathered} A = \mathcal{L}, \\ a\alpha_j \geq \xi_{jj}, \\ -(\partial_k(\alpha_k\partial_k a) + \partial_k(a\partial_k\alpha_j) + 2c\alpha_j + 2a\beta)\xi_j^2 + \partial_j\alpha_k\partial_k a\xi_j\xi_k \geq \varepsilon_1\xi_1^2 + \cdots + \varepsilon_N\xi_N^2, \end{gathered} \tag{1.26}$$

$$\partial_k(\alpha_k\partial_k c + \alpha\partial_k\beta) + 2c\beta \geq 0; \tag{1.27}$$

where ξ_{jj}, ξ_j are nonnegative numbers with positive sum;

$$f = \alpha f_1 + f_2, \quad \textit{where } \frac{\partial\alpha}{\partial n} \geq 0 \textit{ on } \Omega, \tag{1.28}$$

and α is given. If $f \in L^2(\Omega)$ and

$$\mathcal{L}f = 0 \quad \textit{in} \ \ \Omega \tag{1.29}$$

in the case (1.26), (1.27)*, then f entering the Dirichlet problem is uniquely determined by the addtional Neumann data $a\dfrac{\partial u}{\partial n}u = g_1$ on $\partial\Omega$.*
In case (1.28)*, f in uniquely identified by the Neumann data if the coefficients of A do not depend on x_N and $c \geq 0$.*

There is a huge literature on inverse problems, the reader can see the Victor Isakov books [Isa06], [Isa90], and the book [ABT11] by Aster, Richard C and Borchers, Brian and Thurber, Clifford Hal. See also, the thesis about recovery of a coefficient in viscoelasticity models [Buh10] by Maya de Buhan, the thesis referent to the Stokes system and its application in respiratory systems [Egl12] by Anne Eggloffe, and the thesis [Bal11] where Andrea Ballerini treats the stability and reconstruction for an immersed body in a fluid. The lectures [Kav02],[Bal12] by O., Kavian and Guillaume Bal, respectively. Finally, some papers in this context are: [Kli92] by Victor Isakov, the works [SU87] by Silvester and Uhlman, [SU13] by Stefanov and Ulhmann, [Uhl99] by Ulhmann, and [KV84] by Robert Kohn and Michael Vogelius. See also, the works of Masahiro Yamamoto and Oleg Yu Imanuvilov [IY01], [IY98], the work [EEK04] by Egger and Klibanov, [Pue11] by Jean Pierre Puel, [DO06] by A. Doudova and A. Osses, [MOR08] by A. Mercado, A. Osses, and L. Rosier, [BY06] by M. Bellassoued and M. Yamamoto.

1.5 Inverse source problems for the Navier-Stokes equations

In presence of an external force $F = F(x,t)$ acting on the model presented in (1.13), it follows that the Navier-Stokes equations for homogeneous incompressible fluids (with suitable boundary conditions and initial data) are:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + y(\nabla\cdot y) + \nabla p = F & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot y = 0 & \text{in} \quad \Omega\times(0,T), \\ +BC & \text{on} \quad \partial\Omega\times(0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{1.30}$$

To the system (1.30) the inverse source problems are divided in two types: the linear case and the nonlinear case. In the linear case, the system (1.30) is called *Stokes system*. Thus, the Stokes system with Dirichlet homogeneous boundary conditions is given by:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + \nabla p = F & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot y = 0 & \text{in} \quad \Omega\times(0,T), \\ y = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{1.31}$$

where $\Omega\subset\mathbb{R}^N$ is an open boundary set and $N = 2,3$. The pioneers in to deal with inverse source problems for the system (1.31) were Imanuvilov and Yamamoto in [IY00]. In this paper the authors proved the *Lipschitz stability* when the force F only depends on space. In fact, the corresponding inequality is:

$$\begin{aligned} \|F\|_{L^2(\Omega)^N} \le C\Big(\|y(\cdot,\theta)\|_{H^2(\Omega)^N} \quad & +\|\nabla p(\cdot,\theta)\|_{L^2(\Omega)^N} \\ & +\|p\|_{H^1(\theta-\delta,\theta+\delta),L^2(\omega))} + \|y\|_{H^1(\theta-\delta,\theta+\delta),L^2(\omega)^N)}\Big), \end{aligned}$$

where $0<\theta<T$, δ is small positive number and ω is an open subset in Ω. The techniques used by them are Carleman inequalities and the Bukhgeim-Klibanov method, which is useful

in order to solve inverse problems (see [Kli13], [FI96b]). This problem is overdetermined because $p(\cdot,\theta)$ is not necessary for the well-posedness in an initial (or boundary) value problem for the Stokes system (1.31). Moreover, the uniqueness is an open problem whether we choose $\theta = 0$, see for instance[Isa90] and [FK64]. The work of Choulli, Imanuvilov, Puel and Yamamoto [CIPY13] is based on Carleman inequalities in order to prove other inequality (see [IY05], [IPY09]). In [CIPY13] the authors have established the Lipschitz stability for the linearized system associated to (1.30), from measurements only of the velocity. In this case, the pressure term disappears with the rotational operator and the source is $F(x,t) = R(x,t)f(x)$, with $R(x,t)$ vector field known and f unknown. More precisely, they found

$$\|f\|_{L^2(\Omega)} \leq C\Big(\|y\|_{H^2(0,T;H^1(\omega)^N)} + \|\nabla\times y(\cdot,\theta)\|_{H^2(\Omega)^N} + \|y(\cdot,\theta)\|_{H^1(\Omega)^N}\Big),$$

with $0<\theta<T$ and different hypothesis over $R(x,t)$ and $\omega \subset \overline{\Omega}$. From a abstract view, the system (1.31) with $T=+\infty$ was studied in [GT11] by G. García and T. Takahashi to obtain a logarithmical stability. The tools in their paper are Carleman estimates and other types of inequalities that arise from null controllability problems for parabolic equations. Roughly speaking, for a source $F(x,t) = \sigma(t)f(x)$ where f is vector valued, it follows

$$\|f\|_{L^2(\Omega)^N} \leq C\left(\frac{\|\partial_t y\|^q_{L^2(0,\tau;L^2(\omega)^N)}}{\log\|\partial_t y\|_{L^2(0,\tau;L^2(\omega)^N)}}\right)^{s/q}, \tag{1.32}$$

where $q \in (1, 1/(1-\varepsilon))$, ε is a small positive number and $0<\tau<T$.

There are other works making reference to inverse problems for similar systems. For instance, the work [Mar15] by Nuno Martins, where the author uses the Brinkman-Stokes system in order to prove the identification for the external source and a divergence source, from boundary data of the stress tensor. A brief description is the following. To the system

$$\begin{cases} (\Delta-\lambda)y_\lambda - \nabla p_\lambda = f & \text{in} \quad \Omega,\\ \nabla\cdot y_\lambda = g & \text{in} \quad \Omega, \quad \lambda\geq 0\\ y_\lambda = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.33}$$

where the constant λ plays the role of the medium's resistance to the flow, the *inverse problem* is to recover the pair of a body force and a divergence source, (f,g), from measurements data over the stress tensor $\sigma(y_\lambda,p_\lambda)$ on the boundary, with (y_λ,p_λ) satisfying (1.33). Then, they define an operator $\Lambda_\lambda : L^2(\Omega)\to H^{-1/2}(\partial\Omega)$ by $\Lambda_\lambda(f,g) := \sigma(y_\lambda,p_\lambda)n|_{\partial\Omega}$, and through Functional and Fourier analysis the identification from several measurements is achieved.

1.6 Contribution of the thesis

In this thesis we deal with two problems, the first one, the *local null controllability* for the Navier–Stokes system with nonlinear Navier-slip boundary conditions. It is very important to say that our result is obtained in the case where the control has one null scalar component. More precisely, the controllability system is the following:

$$\begin{cases} y_t - \nabla\cdot(Dy) + (y,\nabla)y + \nabla p = v\chi_\omega & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot y = 0 & \text{in} \quad \Omega\times(0,T), \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + f(y)_{tg} = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{1.34}$$

where $f:\mathbb{R}^N\to\mathbb{R}^N$ represents the nonlinearity on the boundary condition, $N=2$ or $N=3$, and v is the control acting on a subdomain $\omega\times(0,T)\subset\Omega\times(0,T)$ such that $v_j=0$, for some $j=1,\dots,N$.

The strategy is divided in five steps. The first one, a new regularity result for the associated linear system, i.e., the linear system is the Stokes system with linear Navier–slip conditions. Thus, the solution (y,p) belongs to $L^2(0,T;H^4(\Omega)^N\cap W)\cap H^2(0,T;L^2(\Omega)^N\cap W)\times L^2(0,T;H^3(\Omega))$, with $W=\{u\in H^1(\Omega)^N : \nabla\cdot y = 0 \text{ in } \Omega,\ \ y\cdot n = 0 \text{ on } \partial\Omega\}$. The second one, a new Carleman estimate in order to prove the null controllability of the linear system, where the pressure term is considered in the estimates. The third, the null controllability for the linear system with control v in $L^2(0,T;H^2(\omega)^N)\cap H^1(0,T;L^2(\omega)^N)$, and of course $v_j=0$, for some $j=1,\dots,N$. The fourth step is to apply Katutani's fixed point theorem in order to prove the null controllability for the Stokes system with nonlinear boundary conditions. Finally, the Implicit mapping theorem allows us to complete the proof. This allow to obtain the local null controllability of (2.1) with internal controls having one vanishing component.

The second part of this thesis treat inverse source problems for the Stokes system with homogeneous Dirichlet boundary conditions from velocity measurements with one missing component. Here, it is important to say that at the moment, the inverse source problem for the system (1.30) with Dirichlet or linear Navier-slip boundary conditions remains open, even if the source only depends on space.
Our main results have two types of sources: $F(x,t)=\sigma(t)f(x)$ and $F(x)=R(x)f(x)$ for (x,t) in $\Omega\times(0,T)$, where $\sigma(t), R(x)$ are known, and in both cases $f(x)$ is unknown, however, in the first case $f(x)$ is vector valued, meanwhile $f(x)$ is scalar in the second case. Then, the inverse source problems obtained for the system (1.31) are: *Reconstruction and uniqueness* for $F(x,t)=\sigma(t)f(x)$ in the space

$$H := \{f\in L^2(\Omega)^N : \nabla\cdot f = 0 \text{ in } \Omega,\ \ f=0 \text{ on } \partial\Omega\},$$

from local measurements of $N-1$ scalar components of velocity, or in other words, the observed data have one missing component of velocity.
Roughly speaking, the reconstruction formula is:

$$P_H f_k = a_k^{-1}(\mathcal{C}_{1k}+\mathcal{C}_{2k}),$$

where P_H represents the orthogonal projector from $L^2(\Omega)^N$ onto H and

$$a_k := 1 - \frac{\nu\lambda_k}{\sigma(T)}\int_0^T \mathrm{e}^{-\nu\lambda_k(T-s)}\sigma(s)\mathrm{d}s \neq 0,$$

for $k \geq 0$, λ_k are eigenvalues of the Stokes operator (with homogeneous Dirichlet boundary conditions) and the functions $\mathcal{C}_{1k}$, $\mathcal{C}_{2k}$ only depend on the local observations of $N-1$ components of the solution of (1.31). In consequence, the uniqueness is achieved for f in H. The proof is based in the works of G. C. García, A. Osses and N.Tapia [GOT13] for a reconstruction formula in parabolic equations, and the work [CG09] by J-M. Coron and S. Guerrero about null controllability of the Stokes system with $N-1$ scalar controls. We also establish numerical experiments in order to see the feasibility our results. Finally we comments some open problems in this context.
Lipschitz stability for $F(x) = R(x)f(x)$ when local and boundary measurements are available, with some additional assumptions respect to $R(x)$. The corresponding inequality is given by:

$$\|f\|_{L^2(\Omega)} \leq C\Bigg(\|\Delta^2 y_j(\cdot,\theta)\mathrm{e}^{s\alpha(\cdot,\theta)}\|_{L^2(\Omega)} + \sum_{k=0}^{2} \|(\hat{\xi})^{1/2}\mathrm{e}^{s\hat{\alpha}}\partial_t^k \Delta y_j\|_{L^2(0,T;H^{5/4}(\partial\Omega))} + \sum_{k=0}^{2} \|\xi^{3/2}\mathrm{e}^{s\alpha}\partial_t^k \Delta y_j\|_{L^2(\omega\times(0,T))}\Bigg), \tag{1.35}$$

where $0 < \theta < T$, $\omega \subset \Omega$ an open subset, α and $\hat{\alpha}$ are Carleman weights, and $s > 0$ is sufficiently large. The proof of (1.35) involves the Bukhgeim–Klibanov method based in Carleman inequalities to prove inverse problems (see [Kli81]). The additional tools are: one Carleman estimates obtained in [FCGBGP06] for parabolic equations with Fourier boundary conditions and some results in the theory of degenerate elliptic operators. A similar result can be developed using degenerate Sobolev spaces (see Appendix A).

Chapter 2

Controllability for the Navier-Stokes with Navier-slip boundary conditions

2.1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class C^∞. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. Here, we will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$ and by $n(x)$ the outward unit normal vector to Ω at the point $x \in \partial\Omega$.
Let us consider the controlled Navier-Stokes system with nonlinear Navier slip boundary conditions. Given a nonlinear regular function $f : \mathbb{R}^N \to \mathbb{R}^N$ and an initial state y_0, we consider the following system:

$$\begin{cases} y_t - \nabla \cdot (Dy) + (y, \nabla)y + \nabla p = v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y \cdot n = 0,\ (\sigma(y,p) \cdot n)_{tg} + f(y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $v = v(x,t)$ stands for the control which acts in a arbitrary fixed domain $\omega \times (0, T)$ and χ_ω is a smooth positive function such that $\chi_\omega = 1$ in ω', where $\omega' \Subset \omega$, with ω' an open set. Respect to the boundary conditions, we mention that in 1823, C.L. Navier (see [Nav23]) established a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangential to the surface should be proportional to the rate of strain at the surface. The velocity's component normal to the surface is naturally zero as mass is not able to penetrate an impermeable solid surface [Nav23]. This can be expressed by

$$y \cdot n = 0 \quad \text{and} \quad (\sigma(y,p) \cdot n)_{tg} + k y_{tg} = 0 \quad \text{on } \Sigma,$$

where $\sigma(y,p) := -pId + Dy$ is the stress tensor, D is the symmetrized gradient of y, p is the pressure, Id is the identity matrix, $(\sigma(y,p) \cdot n)_{tg}$ denotes the tangential component of $\sigma(y,p) \cdot n$ and y_{tg} is the tangential velocity along the solid surface and k is a scalar friction function that measures the local viscous coupling between fluid and solid.

Physically a nonzero slip length arises from the unequal wall and fluid densities, the weak wall-fluid interaction, and the high temperature. These behavior types had been recently demonstrated and showed that the phenomenon of slip occurs with dependence on various factors, such as in aerodynamics processes when the high pressure is involved, in weather forecast where trees, buildings, water waves have to be taken into account, in turbulence, when k depends on pressure, etc. In consequence, the analysis is complicated as well as numerical solutions of the model and an alternative is then to reduce the no-slip condition on rough boundaries to *ad hoc* boundary conditions, the so-called *wall laws*, on a smooth domain.
Let us point out that our boundary conditions corresponds to a law of the wall that appear in turbulent flows, specifically when k may not depend on $|y|$ linearly. We invite to interested reader to see [Bre12],[LBS07] for a complete discussion on this subject.

In the context of controllability, the papers by Coron [Cor96] and Imanuvilov [Ima97] show results of the approximate controllability and local exact controllability for the Navier-Stokes system with Navier-slip boundary conditions in two dimensions, with some restrictions respectively. The system (2.1) has been studied by Guerrero [Gue06], in this paper the author proved the local null controllability to the trajectories of (2.1) in dimension N using Carleman estimates for the associated linear system and fixed point arguments. On the other hand, recent papers by Coron and Guerrero [CG09], Carreño and Guerrero [CG13] are evidence of the null controllability and local null controllability of the Navier-Stokes system with $N-1$ scalar controls, even thought they use homogeneous Dirichlet boundary conditions. Then, the main objective of this Chapter is to obtain the local null controllability of system (2.1) by means of $N-1$ scalar controls, see Theorem 2.1.

Let us now introduce several spaces which are usual in the context of problems modeling incompressible fluids:

$$V := \{u \in H_0^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega\},$$

$$H := \{u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega \;\; u \cdot n = 0 \text{ on } \partial\Omega\}$$

and

$$W = \{u \in H^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \;\; u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Our main result is given in the following theorem.

Theorem 2.1 *Let us assume that $i \in \{1, \ldots, N\}$ and $f \in C^4(\mathbb{R}^N; \mathbb{R}^N)$ with $f(0) = 0$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $y_0 \in H^3(\Omega)^N \cap W$ satisfying $\|y_0\|_{H^3(\Omega)^N \cap W} \leq \delta$ and the compatibility condition*

$$(Dy_0 \cdot n)_{tg} + (f(y_0))_{tg} = 0 \text{ on } \partial\Omega, \tag{2.2}$$

we can find a control

$$v \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N),$$

with $v_i \equiv 0$ and an associated solution (y, p) to (2.1) verifying $y(\cdot, T) = 0$ in Ω.

To prove Theorem 2.1, we first deduce a null controllability result for a linearized system around zero associated to (2.1):

$$\begin{cases} y_t - \nabla\cdot(Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (A(x,t)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{2.3}$$

where A is a $N\times N$ matrix-valued function in a suitable space and h decreases exponentially to zero in T. Finally, we apply Kakutani's fixed point theorem and an inverse mapping theorem to conclude the local null controllability for the nonlinear system (2.1).

On the other hand, we highlight that some ideas as appear in [CG13] and [CG09] concerning to null controllability for the linear system (2.3) are not able to be considered. Indeed, this relevant detail arises from the different boundary conditions that we present here.
The Chapter is organized as follows. In Section 2.2, we present a previous regularity result proved in [Gue06] and other that we prove here for systems as (2.3). In section 2.4 we establish a Carleman inequality needed to deal with the controllability problems. In section 2.4 we prove the null controllability of the linear system (2.3). Finally, in Section 2.5 we give the proof of Theorem 2.1 using fixed point arguments.

Before starting with Section 2, we consider several Hilbert spaces for $\varepsilon > 0$ small enough :

$$P^0_\varepsilon := H^{1/2+\varepsilon}(0,T;H^{1+\varepsilon}(\partial\Omega)^{N\times N}), \quad P^1_\varepsilon := H^{5/4+\varepsilon}(0,T;L^2(\partial\Omega)^{N\times N}),$$
$$P^2 := L^2(0,T;H^{5/2}(\partial\Omega)^{N\times N}),$$
$$Z_\varepsilon := H^{5/4+\varepsilon}(0,T;H^1(\Omega)^N\cap W)\cap L^2(0,T;H^3(\Omega)^N\cap W) \tag{2.4}$$

and

$$Y_1 := L^2(0,T;H^2(\Omega)^N)\cap H^1(0,T;L^2(\Omega)^N), \quad Y_2 := L^2(0,T;H^4(\Omega)^N)\cap H^2(0,T;L^2(\Omega)^N).$$

2.2 Preliminary results

In order to prove the main theorem of this Chapter, we introduce some preliminary results which will be used later on. More precisely, we present regularity results concerning the Stokes system with linear Navier-slip boundary conditions.

The proof of the following result can be found in [Gue06].

Lemma 2.2 *Let $A \in P^0_\varepsilon$, $u_0 \in H$, $f_0 \in L^2(0,T;W')$, $f_2 \in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$ and let u be the weak solution of the system*

$$\begin{cases} u_t - \nabla\cdot(Du) + \nabla\theta = f_0 & in \quad Q, \\ \nabla\cdot u = 0 & in \quad Q, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u)_{tg} = f_2 & on \quad \Sigma, \\ u(\cdot,0) = u_0(\cdot) & in \quad \Omega, \end{cases} \tag{2.5}$$

namely, the function u satisfying

$$\begin{cases} \displaystyle\int_\Omega u_t(t)\cdot v\mathrm{d}x + \frac{1}{2}\int_\Omega Du(t) : Dv\mathrm{d}x + \int_{\partial\Omega} Au(t)\cdot v\mathrm{d}\sigma \\ \displaystyle= \int_\Omega f_0(t)\cdot v\mathrm{d}x + \int_{\partial\Omega} f_2(t)\cdot v\mathrm{d}\sigma \quad a.e\ t\in(0,T), \quad \forall v\in W, \\ u(\cdot,0) = u_0(\cdot) \quad in\ \ \Omega. \end{cases}$$

Then, if we further assume $u_0 \in W$ and

$$f_0 \in L^2(Q)^N,\ f_2 \in L^2(0,T;H^{1/2}(\partial\Omega)^N),\ f_2 \in H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N),$$

u is actually, together with a pressure θ, the strong solution of (2.5), i.e., $(u,\theta) \in Y_1 \times L^2(0,T;H^1(\Omega))$. Furthermore, there exists a positive constant C such that

$$\begin{aligned} \|u\|_{Y_1} + \|\theta\|_{L^2(0,T;H^1(\Omega))} \le C\mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\big(\|f_0\|_{L^2(Q)^N} \\ +\|f_2\|_{L^2(0,T;H^{1/2}(\partial\Omega)^N)} + \|f_2\|_{H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N)} + \|u_0\|_{H^1(\Omega)^N}\big)\,. \end{aligned} \tag{2.6}$$

Remark 2.1 *The author in [Gue06] proved Lemma 2.2 whenever*

$$A \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^{N\times N}),$$

where $0 < \ell < 1/2$ is arbitrarily close to $1/2$ and $\nu_1 > 1$ is arbitrarily close to 1. Observe that this hypothesis is satisfied if $A \in P^0_\varepsilon$.

Using the above Lemma, we prove now a regularity result for the solution of (2.5). To this end, we will impose the following compatibility condition :

$$(Du_0\cdot n)_{tg} + (A(\cdot,0)u_0)_{tg} = f_2(\cdot,0) \quad \text{ on } \partial\Omega. \tag{2.7}$$

Theorem 2.3 *Let $A \in P^1_\varepsilon \cap P^2$, $u_0 \in H^3(\Omega)^N \cap W$ satisfying (2.7), $f_0 \in Y_1$, $f_2 \in L^2(0,T;H^{5/2}(\partial\Omega)^N) \cap H^1(0,T;H^{1/2}(\partial\Omega)^N)$, and let u be the strong solution of system*

$$\begin{cases} u_t - \nabla\cdot(Du) + \nabla\theta = f_0 & in \quad Q, \\ \nabla\cdot u = 0 & in \quad Q, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u\cdot n)_{tg} = f_2 & on \quad \Sigma, \\ u(\cdot,0) = u_0(\cdot) & in \quad \Omega. \end{cases} \tag{2.8}$$

Then, $(u,\theta) \in Y_2 \times L^2(0,T;H^3(\Omega))$ and there exists a positive constant C such that

$$\begin{aligned} &\|u\|_{Y_2} + \|\theta\|_{L^2(0,T;H^3(\Omega))} \\ &\le C(A)\Big(\|f_0\|_{Y_1} + \|f_2\|_{L^2(0,T;H^{5/2}(\partial\Omega)^N)} + \|f_2\|_{H^1(0,T;H^{1/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N}\Big), \end{aligned} \tag{2.9}$$

where

$$C(A) = C\mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1+\|A\|^2_{P^0_\varepsilon}\Big)\Big[1+\|A\|^3_{P^1_\varepsilon} + \|A\|^3_{P^2}\Big]. \tag{2.10}$$

Proof of Theorem 2.3. We consider (2.8) like a parametrized stationary system, that is to say:

$$\begin{cases} -\nabla\cdot(Du)+\nabla\theta = f_0 - u_t & \text{in} \quad \Omega, \\ \nabla\cdot u = 0 & \text{in} \quad \Omega, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u\cdot n)_{tg} = f_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.11}$$

for almost every $t\in(0,T)$.
The rest of the proof is divided in two steps.
Step 1. The goal will be to prove that the weak solution (u,θ) of the stationary system

$$\begin{cases} -\nabla\cdot(Du)+\nabla\theta = g_0 & \text{in} \quad \Omega, \\ \nabla\cdot u = g_1 & \text{in} \quad \Omega, \\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} = g_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.12}$$

actually belongs to $H^3(\Omega)^N\times H^2(\Omega)$, whenever $g_0\in H^1(\Omega)^N$, $g_1\in H^2(\Omega)$ and $g_2\in H^{3/2}(\partial\Omega)^N$.

In accordance with estimate (2.6) for the stationary case and for $A=0$, we obtain that the weak solution of (2.12) satisfies

$$\|u\|_{H^2(\Omega)^N}+\|\theta\|_{H^1(\Omega)}\le C\Big(\|g_0\|_{L^2(\Omega)^N}+\|g_1\|_{H^1(\Omega)}+\|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big), \tag{2.13}$$

for a positive constant C.
The interior regularity readily follows from the corresponding result with homogeneous Dirichlet boundary conditions, which can be found in [Tem01], for instance. Then, for every $\Omega'\subset\subset\Omega$, we have $u\in H^3(\Omega')^N$, $\theta\in H^2(\Omega')$ and

$$\|u\|_{H^3(\Omega')^N}+\|\theta\|_{H^2(\Omega')}\le C\Big(\|g_0\|_{H^1(\Omega)^N}+\|g_1\|_{H^2(\Omega)}+\|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big), \tag{2.14}$$

for some positive constant $C(\Omega',\Omega)$.

We consider $x_0\in\partial\Omega$ and U_0 a simply connected neighborhood of x_0. Then, it suffices to prove that $u\in H^3(\Omega\cap\tilde U)^N$ and $\theta\in H^2(\Omega\cap\tilde U)$, for every $\tilde U\subset\subset U_0$.
To this end, let ψ be a $W^{3,\infty}$ diffeomorphism which sends the set

$$C_0 := \{(\xi',\xi_N)\in\mathbb{R}^N : |\xi_{\mathrm{i}}|<\alpha_0 \quad \mathrm{i}=1,\cdots,N-1, |\xi_N|<\beta_0\}$$

onto U_0 and which verifies

$$\psi(C_0^+)=\Omega\cap U_0,\quad \psi(\Delta_{\alpha_0})=\partial\Omega\cap U_0,$$

where we have denoted $C_0^+ = C_0\cap\mathbb{R}^N_+$ and $\Delta_{\alpha_0}=\partial\mathbb{R}^N_+\cap C_0$. Let us now introduce a cut-off function $\zeta\in C^2(U_0)$ such that

$$\zeta\equiv 1 \text{ in } \tilde U \quad \text{and} \quad \operatorname{supp}\zeta\subset U_1\subset\subset U_0, \tag{2.15}$$

where U_1 is a regular open set. Then, let us set $z=\zeta u$, $h=\zeta\theta$. They verify:

$$\begin{cases} -\nabla\cdot(Dz)+\nabla h = g_0^* & \text{in} \quad \Omega\cap U_0, \\ \nabla\cdot z = g_1^* & \text{in} \quad \Omega\cap U_0, \\ z\cdot n = 0,\ (\sigma(z,h)\cdot n)_{tg} = g_2^* & \text{on} \quad \partial\Omega\cap U_0, \\ z = 0 & \text{on} \quad \Omega\cap\partial U_0, \end{cases} \tag{2.16}$$

with

$$g_0^* = \zeta g_0 - 2\nabla\zeta\cdot\nabla u - \nabla\zeta\cdot\nabla^t u - \Delta\zeta u - \nabla\nabla\zeta\cdot u + \theta\nabla\zeta - g_1\nabla\zeta \in H^1(\Omega\cap U_0)^N,$$
$$g_1^* = \zeta g_1 + \nabla\zeta\cdot u \in H^2(\Omega\cap U_0) \quad \text{and} \quad g_2^* = \zeta g_2 + \frac{\partial\zeta}{\partial n}u \in H^{3/2}(\partial\Omega\cap U_0)^N. \tag{2.17}$$

Let us now perform the change of variable $x = \psi(\xi)$. If we define $\tilde{z} = z\circ\psi$, $\tilde{h} = h\circ\psi$ and $\tilde{n} = n\circ\psi$, then

$$\frac{\partial}{\partial x_{\mathrm{i}}} z_s = \sum_{k=1}^N \frac{\partial \tilde{z}_s}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_{\mathrm{i}}} = \nabla\tilde{z}_s\cdot\nabla_{\mathrm{i}}\psi^{-1}, \quad \forall s = 1,\ldots,N,$$

where we have denoted $\nabla_{\mathrm{i}}\psi^{-1}$ the ith-column of $\nabla\psi^{-1}$. Observe that

$$\frac{\partial}{\partial x_l}\Big(\frac{\partial}{\partial x_{\mathrm{i}}} z_s\Big) = \sum_{j,k=1}^N \Big(\frac{\partial^2\tilde{z}_s}{\partial\xi_k\partial\xi_j}\frac{\partial\xi_j}{\partial x_l}\frac{\partial\xi_k}{\partial x_{\mathrm{i}}}\Big) + \sum_{k=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\frac{\partial^2\xi_k}{\partial x_l\partial x_{\mathrm{i}}}.$$

Therefore

$$\begin{aligned}\Delta z_s &= \sum_{\mathrm{i},j,k=1}^N \Big(\frac{\partial^2\tilde{z}_s}{\partial\xi_k\partial\xi_j}\frac{\partial\xi_j}{\partial x_{\mathrm{i}}}\frac{\partial\xi_k}{\partial x_{\mathrm{i}}}\Big) + \sum_{k,\mathrm{i}=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\frac{\partial^2\xi_k}{\partial x_{\mathrm{i}}^2} = Hess(\tilde{z}_s) : \Big(\sum_{\mathrm{i}=1}^N \frac{\partial\xi_j}{\partial x_{\mathrm{i}}}\frac{\partial\xi_k}{\partial x_{\mathrm{i}}}\Big)_{j,k} + \sum_{k=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_k}\Delta\xi_k \\ &= Hess(\tilde{z}_s) : \nabla\psi^{-1}\nabla^t\psi^{-1} + \nabla\tilde{z}_s\cdot\Delta\psi^{-1},\end{aligned}$$

where $Hess(\tilde{z}_s)$ represents the Hessian matrix on $\tilde{z}_s$ and $\Delta\psi^{-1} := \Delta\xi := (\Delta\xi_1,\ldots,\Delta\xi_N)$. Moreover,

$$\mathrm{div}\, z = \sum_{s,j=1}^N \frac{\partial\tilde{z}_s}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_s} = \nabla\tilde{z} : \nabla^t\psi^{-1} \quad \text{and} \quad \frac{\partial}{\partial x_s}h = \sum_{j=1}^N \frac{\partial\tilde{h}}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_s} = \nabla\tilde{h}\cdot\nabla_s\psi^{-1}.$$

Then, taking into account that for every $\mathrm{i} = 1,\ldots,N$ we have

$$(\nabla\cdot Dz)_{\mathrm{i}} = \Delta z_{\mathrm{i}} + \partial_{\mathrm{i}}\,\mathrm{div}\, z,$$

we find from (2.16) that $\tilde{z}_{\mathrm{i}}$ satisfies the following system for $\mathrm{i} = 1,\ldots,N$:

$$\begin{cases} -Hess(\tilde{z}_{\mathrm{i}}) : \nabla\psi^{-1}\nabla^t\psi^{-1} - \nabla\tilde{z}_{\mathrm{i}}\cdot\Delta\psi^{-1} + \nabla\tilde{h}\cdot\nabla_{\mathrm{i}}\psi^{-1} = (\tilde{g_0}^*)_{\mathrm{i}} + \partial_{\mathrm{i}}\tilde{g_1}^* & \text{in} \quad C_0^+, \\ \nabla\tilde{z} : \nabla^t\psi^{-1} = \tilde{g_1}^* & \text{in} \quad C_0^+, \\ \tilde{z}\cdot\tilde{n} = 0, \quad (\tilde{\sigma}(\tilde{z})\cdot\tilde{n})_{tg} = \tilde{g_2}^* & \text{on} \quad \partial\mathbb{R}^N_+\cap C_0, \\ \tilde{z} = 0 & \text{on} \quad \partial C_0^+\cap\mathbb{R}^N_+, \end{cases} \tag{2.18}$$

where we have denoted

$$\tilde{g_0}^* = g_0^*\circ\psi, \quad \tilde{g_1}^* = g_1^*\circ\psi, \quad \tilde{g_2}^* = g_2^*\circ\psi$$

and

$$(\tilde{\sigma}(\tilde{z}))_{\mathrm{i}s} := \nabla\tilde{z}_s\cdot\nabla_{\mathrm{i}}\psi^{-1} + \nabla\tilde{z}_{\mathrm{i}}\cdot\nabla_s\psi^{-1}, \ \forall 1\le \mathrm{i}, s\le N.$$

On the other hand, note that for every function F in $H^\ell(\Omega)$ ($\ell\in\mathbb{N}, \ell\le 3$), $\tilde{F} = F\circ\psi$ belongs to $H^\ell(C_0^+)$ and there exists a positive constant $C = C(\Omega)$ such that

$$\|\tilde{F}\|_{H^\ell(C_0^+)} \le C\|F\|_{H^\ell(\Omega)}.$$

Now, observe that $\tilde{z} \in \tilde{X}_{0,2}$, with

$$\tilde{X}_{0,2} := \{\tilde{z} \in H^2(C_0^+)^N : \tilde{z} = 0 \text{ on } \partial C_0^+ \cap \mathbb{R}^N_+,\ \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_0\}.$$

Let us introduce $C_1 = \psi(U_1)$ (recall that $U_1 \subset\subset U_0$) and $\mathrm{d} = \mathrm{dist}(\partial C_0^+, \partial C_1^+)$. Then, we have $\delta_m^k \tilde{z} \in \tilde{X}_{0,2}$ for any $1 \le k \le N-1$ and any $|m| < \mathrm{d}/2$, where we have denoted

$$\tilde{X}_{1,2} := \{\tilde{z} \in H^2(C_1^+)^N : \tilde{z} = 0 \text{ on } \partial C_1^+ \cap \mathbb{R}^N_+,\ \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_1\},$$

and

$$\delta_m^k(f) := \tau_m^k(f) - f, \quad \tau_m^k(f) = (\xi \to f(\xi + m\mathrm{e}_k)) \tag{2.19}$$

(see (2.13) and (2.15)). We denote now $\tilde{w} = \delta_m^k \tilde{z}$, $\tilde{\pi} = \delta_m^k \tilde{h}$. We have :

$$\delta_m^k(Hess(\tilde{z}_{\mathrm{i}}) : \nabla\psi^{-1}\nabla^t\psi^{-1}) = Hess(\tilde{w}_{\mathrm{i}}) : \nabla\psi^{-1}\nabla^t\psi^{-1} + Hess(\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k)) : \delta_m^k(\nabla\psi^{-1}\nabla^t\psi^{-1}).$$

$$\delta_m^k(\nabla\tilde{z}_{\mathrm{i}} \cdot \Delta\psi^{-1}) = \nabla\tilde{w}_{\mathrm{i}} \cdot \Delta\psi^{-1} + \nabla\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k) \cdot \delta_m^k(\Delta\psi^{-1}).$$

$$\delta_m^k(\nabla\tilde{z} : \nabla^t\psi^{-1}) = \nabla\tilde{w} : \nabla^t\psi^{-1} + \nabla\tilde{z}(\xi + m\mathrm{e}_k) : \delta_m^k\nabla^t\psi^{-1}$$

and

$$\delta_m^k(\nabla\tilde{h} \cdot \nabla_{\mathrm{i}}\psi^{-1}) = \nabla\tilde{\pi} \cdot \nabla_{\mathrm{i}}\psi^{-1} + \nabla\tilde{h}(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_{\mathrm{i}}\psi^{-1}.$$

Additionally,

$$\delta_m^k(\tilde{z} \cdot \tilde{n}) = \tilde{w} \cdot \tilde{n}$$

and

$$\delta_m^k((\sigma(\tilde{z}, \tilde{h}) \cdot \tilde{n})_{tg}) = (\tilde{\sigma}(\tilde{w}) \cdot \tilde{n})_{tg} + \left[\sum_{s=1}^N (\nabla\tilde{z}_s(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_{\mathrm{i}}\psi^{-1} + \nabla\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_s\psi^{-1})\tilde{n}_s\right]_{tg}$$

on $\partial\mathbb{R}^N_+ \cap C_1$. The last two identities readily follow from (2.19) and the fact that $\tilde{n}_j(\xi + m\mathrm{e}_k) = \tilde{n}_j(\xi)$ on $C_1 \cap \partial\mathbb{R}^N_+$, for every $k = 1, \ldots, N-1$ and for every $j = 1, \ldots, N$. Taking into account the above identities and (2.18), the pair $(\tilde{w}, \tilde{\pi})$ satisfies:

$$\begin{cases} -Hess(\tilde{w}_{\mathrm{i}}) : \nabla\psi^{-1}\nabla^t\psi^{-1} - \nabla\tilde{w}_{\mathrm{i}} \cdot \Delta\psi^{-1} + \nabla\tilde{\pi} \cdot \nabla_{\mathrm{i}}\psi^{-1} = G_{0,\mathrm{i}} + \partial_{\mathrm{i}} G_1 & \text{in} \quad C_1^+, \\ \nabla\tilde{w} : \nabla^t\psi^{-1} = G_1 & \text{in} \quad C_1^+, \\ \tilde{w} \cdot \tilde{n} = 0, \quad (\tilde{\sigma}(\tilde{w}) \cdot \tilde{n})_{tg} = G_2 & \text{on} \quad \partial\mathbb{R}^N_+ \cap C_1, \end{cases} \tag{2.20}$$

where

$$\begin{aligned} G_{0,\mathrm{i}} =& \delta_m^k(\tilde{g_0}^*)_{\mathrm{i}} + Hess(\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k)) : \delta_m^k(\nabla\psi^{-1}\nabla^t\psi^{-1}) + \nabla\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k) \cdot \delta_m^k\Delta\psi^{-1} \\ &+ \partial_{\mathrm{i}}(\nabla\tilde{z}(\xi + m\mathrm{e}_k) : \delta_m^k\nabla^t\psi^{-1}) - \nabla\tilde{h}(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_{\mathrm{i}}\psi^{-1}, \\ G_1 =& \delta_m^k(\tilde{g_1}^*) - \nabla\tilde{z}(\xi + m\mathrm{e}_k) : \delta_m^k\nabla^t\psi^{-1}, \\ G_2 =& \delta_m^k(\tilde{g_2}^*) - \left[\sum_{s=1}^N (\nabla\tilde{z}_s(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_{\mathrm{i}}\psi^{-1} + \nabla\tilde{z}_{\mathrm{i}}(\xi + m\mathrm{e}_k) \cdot \delta_m^k\nabla_s\psi^{-1})\tilde{n}_s\right]_{tg}. \end{aligned}$$

Let us now estimate $G_{0,\mathrm{i}}$ in the $L^2(C_1^+)$ norm. We have

$$\|\delta_m^k(\tilde{g_0}^*)_{\mathrm{i}}\|_{L^2(C_1^+)} \le C|m|\|\nabla(\tilde{g_0}^*)_{\mathrm{i}}\|_{L^2(C_1^+)} \le C|m|\|(\tilde{g_0}^*)_{\mathrm{i}}\|_{H^1(C_1^+)},$$

$$\|Hess(\tilde{z}_{\rm i}(\xi+m{\rm e}_k)) : \delta_m^k(\nabla\psi^{-1}\nabla^t\psi^{-1})\|_{L^2(C_1^+)} \le C|m|\|\tilde{z}\|_{H^2(C_1^+)^N},$$
$$\|\nabla\tilde{z}_{\rm i}(\xi+m{\rm e}_k)\cdot\delta_m^k\Delta\psi^{-1}\|_{L^2(C_1^+)} \le C(k,\Omega)|m|\|\nabla\tilde{z}_{\rm i}\|_{L^2(C_1^+)^N},$$
$$\|\partial_{\rm i}(\nabla\tilde{z}(\xi+m{\rm e}_k) : \delta_m^k\nabla^t\psi^{-1})\|_{L^2(C_1^+)} \le C|m|\|\tilde{z}\|_{H^2(C_1^+)^N}$$

and

$$\|\nabla\tilde{h}(\xi+m{\rm e}_k)\cdot\delta_m^k\nabla_{\rm i}\psi^{-1}\|_{L^2(C_1^+)} \le C|m|\|\nabla\tilde{h}\|_{L^2(C_1^+)^N}.$$

Therefore

$$\|G_0\|_{L^2(C_1^+)^N} \le C|m|\Big(\|g_0^*\|_{H^1(\Omega)^N} + \|z\|_{H^2(\Omega)^N} + \|\nabla h\|_{L^2(\Omega)}\Big).$$

In the same way we can estimate G_1 in $H^1(C_1^+)$ from

$$\|\delta_m^k(\tilde{g_1}^*)\|_{H^1(C_1^+)} \le |m|\|\tilde{g_1}^*\|_{H^2(C_1^+)}$$

and we obtain

$$\|G_1\|_{H^1(C_1^+)} \le C|m|\Big(\|g_1^*\|_{H^2(\Omega)} + \|z\|_{H^2(\Omega)^N}\Big).$$

Finally, for G_2 we get

$$\|G_2\|_{H^{1/2}(\partial\mathbb{R}_+^N\cap C_1)^N} \le C|m|(\|\tilde{g}_2^*\|_{H^{3/2}(\partial\mathbb{R}_+^N\cap C_1)^N} + \|\tilde{z}\|_{H^{3/2}(\partial\mathbb{R}_+^N\cap C_1)^N}).$$

Then, using the definition of $g_{\rm i}^*$ ($\rm i = 0,1,2$) given in (2.17) and the estimate (2.13) for the solutions of the stationary problems (2.12) and (2.16), we obtain

$$\|G_0\|_{L^2(C_1^+)^N} \le C|m|\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^1(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big),$$

$$\|G_1\|_{H^1(C_1^+)} \le C|m|\Big(\|g_0\|_{L^2(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{1/2}(\partial\Omega)^N}\Big)$$

and

$$\|G_2\|_{H^{1/2}(\partial\mathbb{R}_+^N\cap C_1)^N} \le C|m|\Big(\|g_2\|_{H^{3/2}(\partial\Omega)^N} + \|g_0\|_{L^2(\Omega)^N} + \|g_1\|_{H^1(\Omega)}\Big).$$

In consequence, the solution of (2.20) belongs to $\tilde{X}_{1,2}\times H^1(C_1^+)$ and satisfies

$$\|\delta_m^k\tilde{z}\|_{H^2(C_1^+)^N} + \|\delta_m^k\tilde{h}\|_{H^1(C_1^+)} \le C|m|\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N}\Big)$$

for $k = 1,\dots,N-1$. Taking $m\to 0$, this implies $(\partial_k\tilde{z},\partial_k\tilde{h}) \in H^2(C_1^+)^N\times H^1(C_1^+)$ and

$$\|\partial_k\tilde{z}\|_{H^2(C_1^+)} + \|\partial_k\tilde{h}\|_{H^1(C_1^+)} \le C(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N})$$

for $1\le k\le N-1$. Now, we will prove that $\Big(\dfrac{\partial\tilde{z}_{\rm i}}{\partial\xi_N},\dfrac{\partial\tilde{h}}{\partial\xi_N}\Big) \in H^2(C_1^+)\times H^1(C_1^+)$ for every $\rm i = 1,\dots,N$.
From (2.18) we have

$$-\frac{\partial^2\tilde{z}_{\rm i}}{\partial\xi_N^2}\sum_{k=1}^N\Big|\frac{\partial\xi_N}{\partial x_k}\Big|^2 + \frac{\partial\tilde{h}}{\partial\xi_N}\frac{\partial\xi_N}{\partial x_{\rm i}} \in H^1(C_1^+),\quad \forall{\rm i}=1,\dots N. \tag{2.21}$$

Then

$$-\left(\sum_{k=1}^N\Big|\frac{\partial\xi_N}{\partial x_k}\Big|^2\right)\left(\sum_{\rm i=1}^N\frac{\partial^3\tilde{z}_{\rm i}}{\partial\xi_N^3}\frac{\partial\xi_N}{\partial x_{\rm i}}\right) + \frac{\partial^2\tilde{h}}{\partial\xi_N^2}\sum_{\rm i=1}^N\Big|\frac{\partial\xi_N}{\partial x_{\rm i}}\Big|^2 \in L^2(C_1^+). \tag{2.22}$$

On the other hand, from the divergence free condition (see (2.18)) we get

$$\sum_{\mathrm{i}=1}^{N} \frac{\partial \tilde{z}_{\mathrm{i}}}{\partial \xi_N} \frac{\partial \xi_N}{\partial x_{\mathrm{i}}} = -\sum_{\mathrm{i}=1}^{N} \left(\sum_{k=1}^{N-1} \frac{\partial \tilde{z}_{\mathrm{i}}}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_{\mathrm{i}}} \right) + \tilde{g_1}^* \in H^2(C_1^+),$$

so that

$$\sum_{\mathrm{i}=1}^{N} \frac{\partial^3 \tilde{z}_{\mathrm{i}}}{\partial \xi_N^3} \frac{\partial \xi_N}{\partial x_{\mathrm{i}}} \in L^2(C_1^+). \tag{2.23}$$

From (2.22) and (2.23), we obtain that

$$\frac{\partial^2 \tilde{h}}{\partial \xi_N^2} \sum_{\mathrm{i}=1}^{N} \left| \frac{\partial \xi_N}{\partial x_{\mathrm{i}}} \right|^2 \in L^2(C_1^+)$$

and therefore $\tilde{h} \in H^2(C_1^+)$. Coming back to (2.21) we obtain that

$$\frac{\partial^3 \tilde{z}_{\mathrm{i}}}{\partial \xi_N^3} \sum_{k=1}^{N} \left| \frac{\partial \xi_N}{\partial x_k} \right|^2 \in L^2(C_1^+), \quad \forall \mathrm{i} = 1, \ldots, N.$$

Therefore $\tilde{h} \in H^2(C_1^+)$ and $\tilde{z} \in H^3(C_1^+)^N$, so that $(\partial_k z, \partial_k h) \in H^2(\Omega \cap \tilde{U})^N \times H^1(\Omega \cap \tilde{U})$ for $k = 1, \ldots, N$ and we can conclude that $(z, h) \in H^3(\Omega \cap \tilde{U})^N \times H^2(\Omega \cap \tilde{U})$ for every $\tilde{U} \subset\subset U$ with the estimate

$$\|z\|_{H^3(\Omega\cap\tilde{U})^N} + \|h\|_{H^2(\Omega\cap\tilde{U})} \leq C\Big(\|g_0\|_{H^1(\Omega\cap\tilde{U})^N} + \|g_1\|_{H^2(\Omega\cap\tilde{U})} + \|g_2\|_{H^{3/2}(\partial\Omega\cap\tilde{U})^N}\Big). \tag{2.24}$$

This, together with (2.14), gives the following estimate for the solution of the stationary system (2.11):

$$\begin{aligned} &\|u\|_{H^3(\Omega)^N} + \|\theta\|_{H^2(\Omega)} \\ &\leq C\Big(\|f_0\|_{H^1(\Omega)^N} + \|u_t\|_{H^1(\Omega)^N} + \|f_2\|_{H^{3/2}(\partial\Omega)^N} + \|Au\|_{H^{3/2}(\partial\Omega)^N}\Big). \end{aligned} \tag{2.25}$$

Now, to estimate the term $\|u_t(t)\|_{H^1(\Omega)^N}$ we multiply (2.8) by

$$\partial_t(B(u,\theta)) := -\nabla \cdot Du_t + \nabla\theta_t$$

and integrate in Ω. We get

$$-\int_\Omega u_t \nabla \cdot Du_t \mathrm{d}x + \int_\Omega u_t \cdot \nabla\theta_t \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |B(u,\theta)|^2 \mathrm{d}x = \int_\Omega f_0 \cdot \nabla\theta_t \mathrm{d}x - \int_\Omega f_0 \nabla \cdot Du_t \mathrm{d}x.$$

Integrating by parts and using that f_0 belongs to W, we obtain

$$\begin{aligned} &\int_\Omega |\nabla u_t|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |B(u,\theta)|^2 \mathrm{d}x - \int_{\partial\Omega} u_t \cdot (Du_t \cdot n)_{tg} \mathrm{d}\sigma \\ &= \int_\Omega \nabla f_0 \cdot \nabla u_t \mathrm{d}x - \int_{\partial\Omega} f_0 \cdot (Du_t \cdot n)_{tg} \mathrm{d}\sigma. \end{aligned}$$

We use now $(Du_t \cdot n)_{tg} = \partial_t f_2 - \partial_t (Au)$:

$$\int_\Omega |\nabla u_t|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |B(u,\theta)|^2 \mathrm{d}x + \int_{\partial\Omega} \partial_t(Au)\cdot u_t \mathrm{d}\sigma$$
$$= \int_\Omega \nabla f_0 \cdot \nabla u_t \mathrm{d}x + \int_{\partial\Omega} (\partial_t f_2)\cdot u_t \mathrm{d}\sigma + \int_{\partial\Omega} \partial_t(Au)\cdot f_0 \mathrm{d}\sigma - \int_{\partial\Omega} \partial_t f_2 \cdot f_0 \mathrm{d}\sigma,$$

for almost every $t \in (0,T)$. Coming back to (2.25), we get

$$\begin{aligned} &\|\nabla u_t\|^2_{L^2(\Omega)^N} + \|u\|^2_{H^3(\Omega)^N} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |B(u,\theta)|^2 \mathrm{d}x + \|\theta\|^2_{H^2(\Omega)} \\ &\le C\Big(\|f_0\|^2_{H^1(\Omega)^N} + \|f_2\|^2_{H^{3/2}(\partial\Omega)^N} + \|Au\|^2_{H^{3/2}(\partial\Omega)^N} + \int_{\partial\Omega} |\partial_t(Au)||u_t|\mathrm{d}\sigma \\ &\quad + \int_{\partial\Omega} |\partial_t(Au)||f_0|\mathrm{d}\sigma + \int_{\partial\Omega} |\partial_t f_2 \cdot u_t|\mathrm{d}\sigma + \int_{\partial\Omega} |\partial_t f_2 \cdot f_0|\mathrm{d}\sigma + \|u_t\|^2_{L^2(\Omega)^N}\Big), \end{aligned} \tag{2.26}$$

for almost every $t \in (0,T)$.
In order to estimate the third term in (2.26) we use that

$$H^{3/2}(\partial\Omega)\cdot H^{3/2}(\partial\Omega) \subset H^{3/2}(\partial\Omega) \quad \text{continuously.}$$

Then

$$\|Au\|^2_{H^{3/2}(\partial\Omega)^N} \le C\|A\|^2_{H^{3/2}(\partial\Omega)^{N\times N}}\|u\|^2_{H^{3/2}(\partial\Omega)^N} \le C\|A\|^2_{H^{3/2}(\partial\Omega)^{N\times N}}\|u\|^2_{H^2(\Omega)^N}.$$

From this estimate and (2.26) we obtain

$$\begin{aligned} &\|\nabla u_t\|^2_{L^2(Q)^N} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))} \\ &\le C\Big(\|f_0\|^2_{L^2(H^1(\Omega)^N)} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})}\|u\|^2_{L^2(H^2(\Omega)^N)} \\ &+ \iint_\Sigma (|\partial_t(Au)| + |\partial_t f_2|)(|u_t| + |f_0|)\mathrm{d}\sigma\mathrm{d}t + \|B(u_0,\theta(0))\|^2_{L^2(\Omega)^N} + \|u_t\|^2_{L^2(Q)^N}\Big), \end{aligned} \tag{2.27}$$

where $\theta(0)$ is defined (up to a constant) by

$$\begin{cases} -\Delta\theta(0)(\cdot) = -\nabla f_0(\cdot,0) & \text{in } \Omega, \\ \dfrac{\partial\theta(0)}{\partial n}(\cdot) = \Delta u_0(\cdot)\cdot n + f_0(\cdot,0)\cdot n & \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

Now, we estimate the boundary terms in (2.27). First, we find

$$\begin{aligned} \iint_\Sigma |\partial_t(Au)|(|u_t| + |f_0|)\mathrm{d}\sigma\mathrm{d}t &\le C_\delta(\|A\|^4_{L^\infty(\Sigma)^{N\times N}}\|u_t\|^2_{L^2(Q)^N} + \|A_t\|^2_{L^2(\Sigma)^{N\times N}}\|u\|^2_{L^\infty(H^1(\Omega)^N)}) \\ &+ \delta(\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|f_0\|^2_{L^2(H^1(\Omega)^N)}) \end{aligned}$$

for any $\delta > 0$. The second term can be estimated as follows :

$$\iint_\Sigma |\partial_t f_2|(|u_t| + |f_0|)\mathrm{d}\sigma\mathrm{d}t \le C_\delta\|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \delta(\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|f_0\|^2_{L^2(H^1(\Omega)^N)}).$$

Putting together these estimates and (2.27) we can deduce

$$\begin{aligned}&\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))}\\ &\le C\Bigg(\|f_0\|^2_{L^2(H^1(\Omega)^N)} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \|B(u_0,\theta(0))\|^2_{L^2(\Omega)^N}\\ &\qquad + \Big(1 + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})} + \|\partial_t A\|^2_{L^2(\Sigma)^{N\times N}} + \|A\|^4_{L^\infty(\Sigma)^{N\times N}}\Big)\|u\|^2_{Y_1}\Bigg).\end{aligned}$$

Using (2.6) in order to estimate $\|u\|^2_{Y_1}$ and elliptic estimates (2.28), we get

$$\begin{aligned}&\|u_t\|^2_{L^2(H^1(\Omega)^N)} + \|u\|^2_{L^2(H^3(\Omega)^N)} + \|B(u,\theta)\|^2_{L^\infty(L^2(\Omega)^N)} + \|\theta\|^2_{L^2(H^2(\Omega))}\\ &\le C(A)\Big(\|f_0\|^2_{Y_1} + \|f_2\|^2_{L^2(H^{3/2}(\partial\Omega)^N)} + \|\partial_t f_2\|^2_{L^2(H^{1/2}(\partial\Omega)^N)} + \|u_0\|^2_{H^3(\Omega)^N}\Big),\end{aligned} \tag{2.29}$$

where

$$C(A) := C\mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}(1 + \|A\|^4_{P^0_\varepsilon})\Big(1 + \|A\|^2_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})} + \|\partial_t A\|^2_{L^2(\Sigma)^{N\times N}} + \|A\|^4_{L^\infty(\Sigma)^{N\times N}}\Big).$$

Step 2. Taking into account the previous step, we will prove that the weak solution (u,θ) of (2.12) belongs to $H^4(\Omega)^N \times H^3(\Omega)$ whenever

$$g_0^* \in H^2(\Omega\cap U_0)^N, \quad g_1^* \in H^3(\Omega\cap U_0), \quad g_2^* \in H^{5/2}(\partial\Omega\cap U_0)^N, \tag{2.30}$$

also, ψ is a $W^{4,\infty}$ diffeomorphism. Here, we define

$$\tilde{X}_{1,3} := \{\tilde{z} \in H^3(C_1^+)^N : \tilde{z} = 0 \text{ on } \partial C_1^+ \cap \mathbb{R}^N_+,\ \tilde{z}\cdot\tilde{n} = 0 \text{ on } \partial\mathbb{R}^N_+ \cap C_1\}.$$

Let us prove that $\tilde{z}$ satisfies $\delta^k_m \tilde{z} \in \tilde{X}_{1,3}$, for $k = 1,\dots,N-1$ and $|m| < \mathrm{d}/2$ (recall that $\mathrm{d} = \mathrm{d}ist(\partial C_0^+, \partial C_1^+)$), where $\tilde{z}$ fulfills (2.18). We have the following estimates for G_0, G_1 and G_2 (which were defined right after (2.20)) :

$$\|G_0\|_{H^1(C_1^+)^N} \le C|m|\Big(\|g_0^*\|_{H^2(\Omega)^N} + \|z\|_{H^3(\Omega)^N} + \|\nabla h\|_{H^1(\Omega)}\Big).$$

$$\|G_1\|_{H^2(C_1^+)} \le C|m|\Big(\|g_1^*\|_{H^3(\Omega)} + \|z\|_{H^3(\Omega)^N}\Big)$$

and

$$\|G_2\|_{H^{3/2}(\partial\mathbb{R}^N_+\cap C_1)^N} \le C|m|(\|\tilde{g}_2^*\|_{H^{5/2}(\partial\mathbb{R}^N_+\cap C_1)^N} + \|\tilde{z}\|_{H^{5/2}(\partial\mathbb{R}^N_+\cap C_1)^N}).$$

Then, using (2.30) together with the definition of g_{i}^* (i $= 0,1,2$) given in (2.17) and the estimate (2.29) for the solutions of the stationary problems (2.12) and (2.16), we obtain

$$\|G_0\|_{H^1(C_1^+)^N} \le C|m|\Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N}\Big),$$

$$\|G_1\|_{H^2(C_1^+)} \le C|m|\Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N}\Big)$$

and

$$\|G_2\|_{H^{3/2}(\partial\mathbb{R}^N_+\cap C_1)^N} \le C|m|\Big(\|g_0\|_{H^1(\Omega)^N} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\Big).$$

In consequence, $(\delta_m^k \tilde{z}, \delta_m^k \tilde{h}) \in \tilde{X}_{1,3} \times H^2(C_1^+)$ and

$$\|\delta_m^k \tilde{z}\|_{H^3(C_1^+)^N} + \|\delta_m^k \tilde{h}\|_{H^2(C_1^+)} \le C|m|\Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\Big)$$

for $k = 1, \ldots, N-1$.

Arguing now as in **Step 1**, we find

$$\|u\|_{H^4(\Omega)^N} + \|h\|_{H^3(\Omega)} \le C\Big(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\Big). \tag{2.31}$$

From (2.31) we obtain the estimate for the solution of the stationary system (2.11):

$$\|u\|_{H^4(\Omega)^N} + \|\theta\|_{H^3(\Omega)} \le C\Big(\|f_0\|_{H^2(\Omega)^N} + \|u_t\|_{H^2(\Omega)^N} + \|f_2\|_{H^{5/2}(\partial\Omega)^N} + \|Au\|_{H^{5/2}(\partial\Omega)^N}\Big), \tag{2.32}$$

for almost every $t \in (0,T)$. Now, in order to estimate the second term of the right-hand side of (2.32), we consider the system satisfied by $(\partial_t u, \partial_t \theta)$ (see (2.8)) :

$$\begin{cases} \partial_t(u_t) - \nabla\cdot(Du_t) + \nabla\theta_t = \partial_t f_0 & \text{in} \quad Q, \\ \nabla\cdot u_t = 0 & \text{in} \quad Q, \\ u_t\cdot n = 0,\ (\sigma(u_t,\theta_t)\cdot n)_{tg} + (Au_t)_{tg} = \partial_t f_2 - (A_t u)_{tg} & \text{on} \quad \Sigma, \\ u_t(\cdot,0) = \nabla\cdot Du_0(\cdot) - \nabla\theta(\cdot,0) + f_0(\cdot,0) & \text{in} \quad \Omega. \end{cases} \tag{2.33}$$

In virtue of Lemma 2.2 we have that (u_t, θ_t) is the strong solution of (2.33). Furthermore, we get $u_t \in Y_1$ and

$$\begin{aligned} \|u_t\|_{Y_1} \le\ & \mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\Big(\|\partial_t f_0\|_{L^2(Q)^N} + \|f_0\|_{L^\infty(H^1(\Omega)^N)} + \|\partial_t f_2\|_{L^2(H^{1/2}(\partial\Omega)^N)} \\ & + \|\partial_t f_2\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} + \|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} \\ & + \|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N\cap W}\Big). \end{aligned} \tag{2.34}$$

Therefore, from (2.32) and (2.34) we obtain

$$\begin{aligned} & \|u_t\|_{Y_1} + \|u\|_{L^2(H^4(\Omega)^N)} + \|\theta\|_{L^2(H^3(\Omega))} \\ & \le \mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}(1+\|A\|^2_{P^0_\varepsilon})\Big(\|f_0\|_{L^2(H^2(\Omega)^N)} + \|\partial_t f_0\|_{L^2(Q)^N} + \|f_2\|_{L^2(H^{5/2}(\partial\Omega)^N)} \\ & \quad + \|\partial_t f_2\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|\partial_t f_2\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} + \|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} \\ & \quad + \|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} + \|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N\cap W}\Big). \end{aligned} \tag{2.35}$$

Finally, we estimate $\|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)}$, $\|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)}$ and $\|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)}$ by:

$$\|A_t u\|_{L^2(H^{1/2}(\partial\Omega)^N)} \le C\|A_t\|_{L^2(H^{1/2}(\partial\Omega)^{N\times N})}\|u\|_{L^\infty(H^2(\Omega)^N)}$$

$$\|A_t u\|_{H^{1/4+\varepsilon}(H^{-\varepsilon}(\Omega)^N)} \le C\|A_t\|_{H^{1/4+\varepsilon}(L^2(\partial\Omega)^{N\times N})}\Big(\|u\|_{L^2(H^3(\Omega)^N)} + \|u\|_{H^1(H^1(\Omega)^N)}\Big)$$

and

$$\|Au\|_{L^2(H^{5/2}(\partial\Omega)^N)} \le C\Big(\|A\|_{L^\infty(H^{3/2}(\partial\Omega)^{N\times N})}\|u\|_{L^2(H^3(\Omega)^N)} + \|A\|_{L^2(H^{5/2}(\partial\Omega)^{N\times N})}\|u\|_{L^\infty(H^2(\Omega)^N)}\Big).$$

Using (2.29), (2.35) and the previous estimates, we find the desired estimate (2.9). This concludes the proof of Theorem 2.3.

2.3 Carleman inequality for the adjoint system

In this section we will prove a Carleman estimate for the adjoint system of (2.3). In order to do so, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega_1 \Subset \omega' \Subset \omega$ and $\eta \in C^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial\Omega.$$

The existence of such a function η is proved in [FI96b]. Then, for all $\lambda \geq 1$ we consider the following weight functions:

$$\begin{aligned}
&\alpha(x,t) = \frac{\mathrm{e}^{2\lambda\|\eta\|_\infty} - \mathrm{e}^{\lambda\eta(x)}}{(t(T-t))^{11}}, \quad && \xi(x,t) = \frac{\mathrm{e}^{\lambda\eta(x)}}{(t(T-t))^{11}}, \\
&\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), && \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t), \\
&\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), && \widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t).
\end{aligned} \tag{2.36}$$

We consider now a backwards non homogeneous system associated to the Stokes equation:

$$\begin{cases}
-\varphi_t - \nabla\cdot(D\varphi) + \nabla\pi = g & \text{in} \quad Q, \\
\nabla\cdot\varphi = 0 & \text{in} \quad Q, \\
\varphi\cdot n = 0,\ (\sigma(\varphi,\pi)\cdot n)_{tg} + (A^t(x,t)\varphi)_{tg} = 0 & \text{on} \quad \Sigma, \\
\varphi(\cdot,T) = \varphi^T(\cdot) & \text{in} \quad \Omega,
\end{cases} \tag{2.37}$$

where $g \in L^2(Q)^N$ and $\varphi^T \in H$. Our Carleman estimate is given in the following proposition.

Proposition 2.4 *Let $A \in P^1_\varepsilon \cap P^2$. There exists a constant λ_0, such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ increasing on $\|A\|_{P^1_\varepsilon \cap P^2}$ and $s_0(\lambda) > 0$ such that for any* $\mathrm{i} \in \{1,\dots,N\}$, *any $g \in L^2(Q)^N$ and any $\varphi^T \in H$, the solution of* (2.37) *satisfies*

$$\begin{aligned}
&s^3 \iint\limits_Q \mathrm{e}^{-6s\alpha^*}(\xi^*)^3|\varphi|^2 \mathrm{d}x\mathrm{d}t \\
&\leq C\Big(\iint\limits_Q \mathrm{e}^{-4s\alpha^*}|g|^2\mathrm{d}x\mathrm{d}t + s^7 \sum_{j=1,j\neq\mathrm{i}}^N \int_0^T\int_{\omega'} \mathrm{e}^{-4s\hat{\alpha}-2s\alpha^*}(\hat{\xi})^{12}|\varphi_j|^2\mathrm{d}x\mathrm{d}t\Big)
\end{aligned} \tag{2.38}$$

for every $s \geq s_0$.

Before giving the proof of Proposition 2.4, we present some technical results. We first present a Carleman inequality proved in [FCGBGP06] for a general heat equation with Fourier boundary conditions. To this end, let us introduce the system

$$\begin{cases}
-\psi_t - \Delta\psi = f_1 + \nabla\cdot f_2 & \text{in} \quad Q, \\
(\nabla\psi + f_2)\cdot n = f_3 & \text{on} \quad \Sigma, \\
\psi(\cdot,T) = \psi^T(\cdot) & \text{in} \quad \Omega,
\end{cases} \tag{2.39}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. We present now this result:

Lemma 2.5 *Under the previous assumptions on f_1, f_2 and f_3, there exist $\overline{\lambda}, \sigma_1, \sigma_2$ and C, only depending on Ω and ω, such that, for any $\lambda \geq \overline{\lambda}$, any $s \geq \overline{s} = \sigma_1(e^{\sigma_2 \lambda} T + T^2)$ and any $\psi^T \in L^2(\Omega)$, the weak solution to (2.39) satisfies*

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla\psi|^2 + s^3\lambda^4\xi^3|\psi|^2)\mathrm{d}x\mathrm{d}t + s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\xi^2|\psi|^2\mathrm{d}\sigma\mathrm{d}t \\
&\qquad \leq C\Big(\iint_Q e^{-2s\alpha}(|f_1|^2 + s^2\lambda^2\xi^2|f_2|^2)\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad + s\lambda \iint_\Sigma e^{-2s\alpha}\xi|f_3|^2\mathrm{d}\sigma\mathrm{d}t + s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2\mathrm{d}x\mathrm{d}t\Big).
\end{aligned} \tag{2.40}$$

The next lemma is a result for elliptic equations with non homogeneous boundary condition that can be found in [IP03] (see also [FCGIP04]).

Lemma 2.6 *Let $y \in H^1(\Omega)$ satisfy*

$$\Delta y = f_0 + \sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j}, \quad \text{in } \Omega\,; \quad y = g, \quad \text{on } \partial\Omega,$$

with $f_0, f_j \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exist three constants $C > 0$, $\hat{\lambda} > 1$ and $\hat{\tau} > 1$ such that for any $\lambda \geq \hat{\lambda}$ and any $\tau \geq \hat{\tau}$, we have

$$\begin{aligned}
&\int_\Omega |\nabla y|^2 e^{2\tau e^{\lambda\eta}}\mathrm{d}x + \tau^2\lambda^2 \int_\Omega e^{2\lambda\eta}|y|^2 e^{2\tau e^{\lambda\eta}}\mathrm{d}x \\
&\quad \leq C\Big(\tau^{1/2}e^{2\tau}\|g\|^2_{H^{1/2}(\partial\Omega)} + \tau^{-1}\lambda^{-2}\int_\Omega e^{-\lambda\eta}|f_0|^2 e^{2\tau e^{\lambda\eta}}\mathrm{d}x \\
&\qquad + \sum_{j=0}^{N} \tau \int_\Omega e^{\lambda\eta}|f_j|^2 e^{2\tau e^{\lambda\eta}}\mathrm{d}x + \int_{\omega_1} (|\nabla y|^2 + \tau^2\lambda^2 e^{2\lambda\eta}|y|^2)e^{2\tau e^{\lambda\eta}}\mathrm{d}x\Big).
\end{aligned} \tag{2.41}$$

Remark 2.2 *We can eliminate the local integral of $|\nabla y|^2$ in (2.41) at the price of having a local term of $|y|^2$ in a open set ω_2 satisfying $\omega_1 \Subset \omega_2 \Subset \omega'$. For these details, we invite the interested reader to see [FCGIP04].*

The next technical result corresponds to the Lemma 3 in [CG09].

Lemma 2.7 *Let $r \in \mathbb{R}$. There exists $C > 0$ depending only on r, Ω, ω_0 and η such that, for every $T > 0$ and every $u \in L^2(0,T; H^1(\Omega))$,*

$$\begin{aligned}
&s^2\lambda^2 \iint_Q e^{-2s\alpha}\xi^{r+2}|u|^2\mathrm{d}x\mathrm{d}t \\
&\qquad \leq C\Big(\iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2\mathrm{d}x\mathrm{d}t + s^2\lambda^2 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^{r+2}|u|^2\mathrm{d}x\mathrm{d}t\Big),
\end{aligned} \tag{2.42}$$

foe every $\lambda \geq C$ and every $s \geq CT^{22}$.

Remark 2.3 *In [CG09], [FCGBGP06] and [IP03] slightly different weight functions are used to prove the above results. However, this does not change the result since the important property is that α goes to 0 polynomially when t tends to 0 and T.*

We will now prove Proposition 2.4. Without any lack of generality, we treat the case $N = 2$ and i $= 2$. The arguments can be easily extended to the general case. Let us introduce (w, q) and (z, r), the solutions of the following systems:

$$\begin{cases} -w_t - \nabla \cdot (Dw) + \nabla q = \rho g & \text{in} \quad Q, \\ \nabla \cdot w = 0 & \text{in} \quad Q, \\ w \cdot n = 0, \, (\sigma(w,q) \cdot n)_{tg} + (A^t(x,t) w)_{tg} = 0 & \text{on} \quad \Sigma, \\ w(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.43}$$

and

$$\begin{cases} -z_t - \nabla \cdot (Dz) + \nabla r = -\rho' \varphi & \text{in} \quad Q, \\ \nabla \cdot z = 0 & \text{in} \quad Q, \\ z \cdot n = 0, \, (\sigma(z,r) \cdot n)_{tg} + (A^t(x,t) z)_{tg} = 0 & \text{on} \quad \Sigma, \\ z(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.44}$$

where $\rho(t) = \mathrm{e}^{-2s\alpha^*}$. Adding (2.43) and (2.44), we see that $(w + z, q + r)$ solves the same system as $(\rho\varphi, \rho\pi)$, where (φ, π) is the solution of (2.37). By uniqueness of the Stokes system with Navier-slip boundary conditions, we have

$$\rho\varphi = w + z \quad \text{and} \quad \rho\pi = q + r. \tag{2.45}$$

For system (2.43) we will use Lemma 2.2 and the regularity estimate (2.6), namely

$$\|w\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|w\|^2_{H^1(0,T;L^2(\Omega)^2)} \leq C \|\rho g\|^2_{L^2(Q)^2}, \tag{2.46}$$

and for the system (2.44) we will use the ideas of [CG13] and [CG09].
We apply the operator ∇ to the equation satisfied by z_1 and we denote $\psi := \nabla z_1$. Then ψ satisfies

$$-\psi_t - \Delta\psi = -\nabla(\rho'\varphi_1) - \nabla\partial_1 r \quad \text{in } Q.$$

Using Lemma 4.1 with $f_1 = -\nabla(\rho'\varphi_1) - \nabla\partial_1 r$ and $f_2 = 0$, we obtain

$$\begin{aligned} s^3 \iint\limits_Q \mathrm{e}^{-2s\alpha} \xi^3 |\psi|^2 \mathrm{d}x\mathrm{d}t \leq C \Big(s^3 \int_0^T \int\limits_{\omega_1} \mathrm{e}^{-2s\alpha} \xi^3 |\psi|^2 \mathrm{d}x\mathrm{d}t \\ + s \iint\limits_\Sigma \mathrm{e}^{-2s\alpha} \xi \Big| \frac{\partial \nabla z_1}{\partial n} \Big|^2 \mathrm{d}\sigma\mathrm{d}t + \iint\limits_Q \mathrm{e}^{-2s\alpha} |\nabla(\rho'\varphi_1) + \nabla\partial_1 r|^2 \mathrm{d}x\mathrm{d}t \Big) \end{aligned} \tag{2.47}$$

for every $\lambda \geq \overline{\lambda}$ and $s \geq \overline{s_0}$. Here and in the following, C will denote a generic constant depending on Ω, ω and λ.
The rest of the proof is divided in three steps.

a) In step 1, using Lemma 2.7 we estimate global integrals of z_1 and z_2. In addition, we partially estimate the pressure in the right-side of (2.47).
b) In step 2, we will estimate the normal derivative appearing in the right-hand side of (2.47) and the global term of the pressure obtained in step 1.
c) In step 3, we will estimate all the local terms by a local term of φ_1.

Step 1. *Estimate of* z_1. We use Lemma 2.7 with $u = \nabla z_1$ and $r = 3$:

$$s^5 \iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2 dx dt \leq C\Big(s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dx dt + s^5 \int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^5|z_1|^2 dx dt\Big) \qquad (2.48)$$

for every $s \geq C$.
Estimate of z_2. Let us first establish a general estimate : $\forall \varepsilon' > 0$, $\exists C \in \mathbb{R}$:

$$\|u\|_{(H^{1/2+\varepsilon'}(\Omega)^2\cap H)'} \leq C(\|u_1\|_{L^2(\Omega)} + \|u_1\, n_1\|_{L^2(\partial\Omega)} + \|\partial_1 u_1\|_{H^{-1/2}(\Omega)}) \leq C\|u_1\|_{H^{1/2+\varepsilon'}(\Omega)},\ \forall u \in W. \qquad (2.49)$$

Indeed, for any $f \in H_{\varepsilon'} := H^{1/2+\varepsilon'}(\Omega)^2 \cap H$, we have (after an integration by parts)

$$\int_\Omega u\cdot f\, dx = \int_\Omega u_1\, f_1\, dx - \int_{\partial\Omega} u_1\, n_1\, \tilde f_2\, d\sigma + \int_\Omega \partial_1 u_1\, \tilde f_2\, dx, \qquad (2.50)$$

where $\tilde f_2 \in H^{1/2+\varepsilon'}(\Omega)$ satisfies

$$\partial_2 \tilde f = f_2 \text{ a. e. } \Omega \quad \textit{and} \quad \|\tilde f_2\|_{H^{1/2+\varepsilon'}(\Omega)} \leq C\|f_2\|_{H^{1/2+\varepsilon'}(\Omega)} \leq C\|f\|_{H_{\varepsilon'}}.$$

Then, from (2.50), we readily obtain (2.49).

Let us now apply (2.49) for $u := z$. We deduce

$$\forall \varepsilon' > 0,\ \exists C \in \mathbb{R} : \|z\|_{(H_{\varepsilon'})'} \leq C\|z_1\|_{H^{1/2+\varepsilon'}(\Omega)},$$

so that, using that $H^{1/2+\varepsilon'}(\Omega)$ is the interpolation space $(H^1(\Omega), L^2(\Omega))_{1/2+\varepsilon',2}$, we find

$$s^{4-2\varepsilon'}\int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'}\|z\|^2_{(H_{\varepsilon'})'} dt \leq Cs^3 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^3\Big(s^2(\xi^*)^2|z_1|^2 + |\nabla z_1|^2\Big) dx dt. \quad (2.51)$$

Putting together (2.47), (2.48) and (2.51) we have for the moment

$$\begin{aligned} s^5 \iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2 dx dt + s^{4-2\varepsilon'}\int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'}\|z\|^2_{(H_{\varepsilon'})'} dt + s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\psi|^2 dx dt \\ \leq C\Big(\int_0^T\int_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2) dx dt \\ + s \iint\limits_\Sigma e^{-2s\alpha}\xi\Big|\frac{\partial \nabla z_1}{\partial n}\Big|^2 d\sigma dt + \iint\limits_Q e^{-2s\alpha}|\nabla(\rho'\varphi_1) + \nabla\partial_1 r|^2 dx dt\Big) \end{aligned} \qquad (2.52)$$

for every $s \geq C$.
Taking into account that

$$|\alpha_t^*| \leq C(\xi^*)^{12/11}, \quad |\rho'| \leq Cs\rho(\xi^*)^{12/11} \tag{2.53}$$

and (2.45), we obtain

$$\begin{aligned}\iint_Q \mathrm{e}^{-2s\alpha}|\nabla(\rho'\varphi_1)|^2\mathrm{d}x\mathrm{d}t =& \iint_Q \mathrm{e}^{-2s\alpha}|\rho'|^2|\rho|^{-2}|\nabla(\rho\varphi_1)|^2\mathrm{d}x\mathrm{d}t \\ \leq& C\Big(s^2\iint_Q \mathrm{e}^{-2s\alpha}(\xi^*)^{24/11}|\nabla w_1|^2 + s^2\iint_Q \mathrm{e}^{-2s\alpha}(\xi^*)^{24/11}|\nabla z_1|^2\mathrm{d}x\mathrm{d}t\Big).\end{aligned} \tag{2.54}$$

The fact that $s^2\mathrm{e}^{-2s\alpha^*}(\xi^*)^{24/11}$ is bounded allows us to estimate the first term in the right-hand side of (2.54) using (2.46). On the other hand, the second term in the right-hand side of (2.54) can be absorbed by the third term in the left-hand side of (2.52).
Additionaly, using the divergence-free condition on the equation of (2.44), we see that

$$\Delta r = 0 \quad \text{in } Q,$$

then

$$\Delta(\nabla\partial_1 r) = 0 \quad \text{in } Q.$$

Using Lemma 2.6 with $y = \nabla\partial_1 r$ and Remark 2.2 we obtain

$$\tau^2\int_\Omega \mathrm{e}^{2\lambda\eta}|\nabla\partial_1 r|^2\mathrm{e}^{2\tau\mathrm{e}^{\lambda\eta}}\mathrm{d}x \leq C\Big(\tau^{1/2}\mathrm{e}^{2\tau}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)} + \tau^2\int_{w_2} \mathrm{e}^{2\lambda\eta}|\nabla\partial_1 r|^2\mathrm{e}^{2\tau\mathrm{e}^{\lambda\eta}}\mathrm{d}x\Big)$$

for every $\tau \geq C$. Now we take

$$\tau = \frac{s}{(t(T-t))11},$$

multiply the last inequality by

$$\exp\Big(-2s\frac{\mathrm{e}^{2\lambda\|\eta\|_\infty}}{(t(T-t))^{11}}\Big),$$

and integrate with respect to t in $(0,T)$ to obtain

$$\begin{aligned}&\iint_Q \mathrm{e}^{-2s\alpha}|\nabla\partial_1 r|^2\mathrm{d}x\mathrm{d}t \\ &\leq C\Big(s^{-3/2}\int_0^T \mathrm{e}^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}\mathrm{d}t + \int_0^T\int_{\omega_2} \mathrm{e}^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2\mathrm{d}x\mathrm{d}t\Big),\end{aligned}$$

for all $s \geq C$.

Combining this with (2.52) and (2.54), we have for the moment

$$
\begin{aligned}
& s^5 \iint\limits_Q e^{-2s\alpha}\xi^5|z_1|^2 dxdt + s^{4-2\varepsilon'} \int_0^T e^{-2s\alpha^*}(\xi^*)^{4-2\varepsilon'} \|z\|^2_{(H_{\varepsilon'})'} dt + s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\nabla z_1|^2 dxdt \\
\leq & C\Big(\|\rho g\|^2_{L^2(Q)^2} + s \iint\limits_\Sigma e^{-2s\alpha}\xi \Big|\frac{\partial \nabla z_1}{\partial n}\Big|^2 d\sigma dt + s^{-3/2} \int\limits_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2} \|\nabla \partial_1 r\|^2_{H^{1/2}(\partial\Omega)} dt \\
& + \int\limits_0^T \int\limits_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2 dxdt + \int\limits_0^T \int\limits_{\omega_1} e^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2) dxdt \Big),
\end{aligned}
\tag{2.55}
$$

for every $s \geq C$.

Step 2. In this step we deal with the boundary terms in (2.55), i.e,

$$
s \iint\limits_\Sigma e^{-2s\alpha}\xi \Big|\frac{\partial \nabla z_1}{\partial n}\Big|^2 d\sigma dt \quad \text{and} \quad s^{-3/2} \int\limits_0^T e^{-2s\alpha^*}(\xi^*)^{-3/2} \|\nabla \partial_1 r\|^2_{H^{1/2}(\partial\Omega)} dt.
$$

Let us start by defining

$$
\check{z} := \check{\theta}(t) z, \quad \check{r} := \check{\theta}(t) r, \quad \check{\theta}(t) := s^{1-\varepsilon'} e^{-s\alpha^*}(\xi^*)^{10/11-\varepsilon'}.
$$

From (2.44), we see that $(\check{z}, \check{r})$ is the solution of the Stokes system:

$$
\begin{cases}
-\check{z}_t - \nabla\cdot(D\check{z}) + \nabla\check{r} = -(\check{\theta})' z - \check{\theta}\rho'\varphi & \text{in} \quad Q, \\
\nabla\cdot\check{z} = 0 & \text{in} \quad Q, \\
\check{z}\cdot n = 0,\ (\sigma(\check{z},\check{r})\cdot n)_{tg} + (A^t(x,t)\check{z})_{tg} = 0 & \text{on} \quad \Sigma, \\
\check{z}(\cdot, T) = 0 & \text{in} \quad \Omega.
\end{cases}
\tag{2.56}
$$

For this system, we have

$$
\begin{aligned}
\|\check{z}\|^2_{L^2(0,T;H^{3/2-\varepsilon'}(\Omega)^2)} & \leq C\Big(\|s^{2-\varepsilon'} e^{-s\alpha^*}(\xi^*)^{2-\varepsilon'} z\|^2_{L^2(0,T;(H_{\varepsilon'})')} + \|s^2 e^{-s\alpha^*}(\xi^*)^2 \rho\varphi\|^2_{L^2(Q)^2} \Big) \\
& \leq C\Big(\|s^{2-\varepsilon'} e^{-s\alpha^*}(\xi^*)^{2-\varepsilon'} z\|^2_{L^2(0,T;(H_{\varepsilon'})')} + \|s^2 e^{-s\alpha^*}(\xi^*)^2 w\|^2_{L^2(Q)^2} \Big).
\end{aligned}
\tag{2.57}
$$

Observe that this inequality comes from Lemma 2.1 with a right-hand side in the interpolation space

$$
(L^2(0,T;W'), L^2(Q))_{1/2+\varepsilon',2} = L^2(0,T;(H_{\varepsilon'})').
$$

The fact that $s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}$ is bounded allows us to use (2.46) and conclude that $\|\check{z}\|^2_{L^2(0,T;H^{3/2-\varepsilon'}(\Omega)^2)}$ is bounded by the left-hand side of (2.55) and $\|\rho g\|^2_{L^2(\Omega)^2}$. Using that

$$
L^2(\Omega)^2 = ((H_{\varepsilon'})', H^{3/2-\varepsilon'}(\Omega)^2)_{3/4-\varepsilon'/2,2},
$$

we deduce that $s^{7/2-3\varepsilon'}\|\mathrm{e}^{-s\alpha^*}(\xi^*)^{7/4-3\varepsilon'/2}z\|^2_{L^2(Q)^2}$ is bounded by the left-hand side of (2.55) and $\|\rho g\|^2_{L^2(\Omega)^2}$. Taking $\varepsilon'>0$ small enough, we deduce in particular that

$$s^3\iint_Q \mathrm{e}^{-2s\alpha^*}(\xi^*)^3|z|^2\mathrm{d}x\,\mathrm{d}t$$

is bounded by the left-hand side of (2.55) and $\|\rho g\|^2_{L^2(\Omega)^2}$.

Next, we define

$$z^*:=\theta^*(t)z,\quad r^*:=\theta^*(t)r,\quad \theta^*(t):=s^{1/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{9/22}.$$

From (2.44), we see that (z^*,r^*) is the solution of (2.56) with $\check{\theta}$ replaced by θ^*. Using again (2.6) and taking into account (2.53), we deduce

$$\begin{aligned}&\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}+\|r^*\|^2_{L^2(0,T;H^1(\Omega))}+\|\theta^* z_t\|^2_{L^2(0,T;L^2(\Omega)^2)}\\ \leq& C\Big(\|s^{3/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2}+\|s^{3/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{3/2}w\|^2_{L^2(Q)^2}\Big).\end{aligned}\tag{2.58}$$

Arguing as before, we conclude that $\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}$ is bounded by the left-hand side of (2.55) and $\|\rho g\|^2_{L^2(\Omega)^2}$.

Now, let

$$\hat{z}:=\hat{\theta}(t)z,\quad \hat{r}:=\hat{\theta}(t)r,\quad \hat{\theta}:=s^{-1/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{-15/22}.$$

From (2.44), $(\hat{z},\hat{r})$ is the solution of (2.56) with $\check{\theta}$ replaced by $\hat{\theta}$. Observe that the right-hand side of this system can be considered in $L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)$ and thus, using the regularity estimate (2.9) we have

$$\begin{aligned}&\|\hat{z}\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^1(0,T;H^2(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)}+\|\hat{r}\|^2_{L^2(0,T;H^3(\Omega))}\\ \leq& C\Big(\|\hat{\theta}'z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}+\|\hat{\theta}\rho'\varphi\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\Big)\\ \leq& C\Big(\|\rho g\|^2_{L^2(Q)^2}+\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)}+\|\theta^* z_t\|^2_{L^2(0,T;L^2(\Omega)^2)}+\|s^{3/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2}\Big).\end{aligned}\tag{2.59}$$

From (2.58), the right-hand side of (2.59) is bounded by

$$\|s^{3/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(Q)^2}\quad\text{and}\quad\|\rho g\|^2_{L^2(Q)^2}.$$

Coming back to (2.55), we find in particular

$$\begin{aligned}&s^5\iint_Q \mathrm{e}^{-2s\alpha}\xi^5|z_1|^2\mathrm{d}x\mathrm{d}t+s^3\iint_Q \mathrm{e}^{-2s\alpha^*}(\xi^*)^3|z_2|^2\mathrm{d}x\mathrm{d}t+\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\\ &\quad+\|\hat{z}\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)}+\|\hat{r}\|^2_{L^2(0,T;H^3(\Omega))}\\ \leq& C\Big(\|\rho g\|^2_{L^2(Q)^2}+s\iint_\Sigma \mathrm{e}^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2\mathrm{d}\sigma\mathrm{d}t+s^{-3/2}\int_0^T \mathrm{e}^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}\mathrm{d}t\\ &\quad+\int_0^T\int_{\omega_2}\mathrm{e}^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\omega_1}\mathrm{e}^{-2s\alpha}(s^5\xi^5|z_1|^2+s^3\xi^3|\nabla z_1|^2)\mathrm{d}x\mathrm{d}t\Big).\end{aligned}\tag{2.60}$$

Observe that the boundary term

$$s^{-3/2}\int_0^T \mathrm{e}^{-2s\alpha^*}(\xi^*)^{-3/2}\|\nabla\partial_1 r\|^2_{H^{1/2}(\partial\Omega)}\mathrm{d}t$$

can be absorbed by the fifth term of the left-hand side of (2.60).
In order to estimate the other boundary term, we notice that α and ξ coincide with α^* and ξ^* respectively on Σ, so that

$$s\iint_\Sigma \mathrm{e}^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2\mathrm{d}\sigma\mathrm{d}t = s\iint_\Sigma \mathrm{e}^{-2s\alpha^*}\xi^*\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2\mathrm{d}\sigma\mathrm{d}t \leq Cs\int_0^T \mathrm{e}^{-2s\alpha^*}\xi^*\|z_1\|^2_{H^{5/2+\varepsilon}(\Omega)}\mathrm{d}t \tag{2.61}$$

for every $\varepsilon > 0$. Taking $\varepsilon = \dfrac{1}{70}$ (any $0<\varepsilon<\frac{1}{70}$ would work) and thanks to an interpolation argument between the spaces $L^2(L^2)$ and $L^2(H^4)$, we obtain

$$\begin{aligned}
s^{43/35}&\int_0^T \mathrm{e}^{-2s\alpha^*}\xi^*\|z_1\|^2_{H^{88/35}(\Omega)}\mathrm{d}t\\
&\leq C\Big(s^5\int_0^T \mathrm{e}^{-2s\alpha^*}(\xi^*)^5\|z_1\|^2_{L^2(\Omega)}\mathrm{d}t + s^{-1}\int_0^T \mathrm{e}^{-2s\alpha^*}(\xi^*)^{-15/11}\|z_1\|^2_{H^4(\Omega)}\mathrm{d}t\Big),
\end{aligned}$$

for every $s \geq C$. Coming back to (2.61) and using the above inequality, the boundary term

$$s\iint_\Sigma \mathrm{e}^{-2s\alpha}\xi\Big|\frac{\partial\nabla z_1}{\partial n}\Big|^2\mathrm{d}\sigma\mathrm{d}t$$

can be absorbed by the left-hand side of (2.60). This ends Step 2.
Thus, at this point we have

$$\begin{aligned}
s^5&\iint_Q \mathrm{e}^{-2s\alpha}\xi^5|z_1|^2\mathrm{d}x\mathrm{d}t + s^3\iint_Q \mathrm{e}^{-2s\alpha^*}(\xi^*)^3|z_2|^2\mathrm{d}x\mathrm{d}t\\
&+\|\hat\theta z\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^2(0,T;L^2(\Omega)^2)} + \|\theta^* z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\\
&\leq C\Big(\|\rho g\|^2_{L^2(Q)^2} + \int_0^T\int_{\omega_2} \mathrm{e}^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\omega_1}\mathrm{e}^{-2s\alpha}(s^5\xi^5|z_1|^2 + s^3\xi^3|\nabla z_1|^2)\mathrm{d}x\mathrm{d}t\Big)
\end{aligned} \tag{2.62}$$

for every $s \geq C$.

Step 3. In this step, we estimate the local term on $\nabla\partial_1 r$ in the right-hand side of (2.62). The other two local terms can be estimated in an easier way.
Let ω_3 be a open subset satisfying $\omega_2 \Subset \omega_3 \Subset \omega'$ and let $\rho_1 \in C^2_c(\omega_3)$ with $\rho_1 \equiv 1$ in ω_2 and $\rho_1 \geq 0$. Then, integrating by parts and using that $\Delta r = 0$ we get

$$\int_0^T\int_{\omega_2} \mathrm{e}^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2\mathrm{d}x\mathrm{d}t \leq C\int_0^T\int_{\omega_3}\Delta(\rho_1\mathrm{e}^{-2s\alpha}\xi^2)|\partial_1 r|^2\mathrm{d}x\mathrm{d}t.$$

From (2.44) and the estimate

$$|\Delta(\rho_1 e^{-2s\alpha}\xi^2)| \le Cs^2 e^{-2s\alpha}\xi^4 1_{\omega_3}, \quad s \ge C,$$

we obtain

$$\int_0^T\int_{\omega_2} e^{-2s\alpha}\xi^2|\nabla\partial_1 r|^2 dx dt \le Cs^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4(|z_{1,t}|^2+|\Delta z_1|^2+|\rho'\varphi_1|^2)dx dt \tag{2.63}$$

for every $s \ge C$. We will now estimate the two first terms in the last integral of (2.63), the third one being estimated in an easier way.

i) Estimate of $z_{1,t}$. We integrate by parts with respect to t:

$$\begin{aligned} s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4|z_{1,t}|^2 dx dt &= \frac{s^2}{2}\int_0^T\int_{\omega_3}\partial_{tt}(e^{-2s\alpha}\xi^4)|z_1|^2 dx dt - s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4 z_{1,tt}z_1 dx dt \\ &\le C\Big(s^4\int_0^T\int_{\omega_3} e^{-2s\alpha}(\xi)^{68/11}|z_1|^2 dx dt + s^2\int_0^T\int_{\omega_3}\hat\theta|z_{1,tt}|\hat\theta^{-1}e^{-2s\alpha}\xi^4|z_1| dx dt\Big), \end{aligned}$$

where we recall that $\hat\theta := s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-15/22}$.
Using Young's inequality for the second term we obtain for every $\varepsilon > 0$

$$\begin{aligned} &s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4|z_{1,t}|^2 dx dt \\ &\le C\Big(s^4\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^7|z_1|^2 dx dt + \varepsilon\int_0^T\int_{\omega_3}|\hat\theta|^2|z_{1,tt}|^2 dx dt + C(\varepsilon)s^5\int_0^T\int_{\omega_3} e^{-4s\alpha+2s\alpha^*}\xi^{10}|z_1|^2 dx dt\Big). \end{aligned} \tag{2.64}$$

The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (2.62).

ii) Estimate of Δz_1. Let w_4 be an open subset such that $w_3 \Subset w_4 \Subset \omega'$ and let $\rho_2 \in C_c^2(w_4)$ with $\rho_2 \equiv 1$ in ω_3 and $\rho_2 \ge 0$. Then, an integration by parts gives

$$\begin{aligned} s^2\int_0^T\int_{\omega_3} e^{-2s\alpha}\xi^4|\Delta z_1|^2 dx dt &\le s^2\int_0^T\int_{\omega_4}\rho_2^2 e^{-2s\alpha}\xi^4|\Delta z_1|^2 dx dt \\ &= -s^2\int_0^T\int_{\omega_4}\nabla(\rho_2^2 e^{-2s\alpha}\xi^4)\cdot\nabla z_1\Delta z_1 dx dt - s^2\int_0^T\int_{\omega_4}\rho_2^2 e^{-2s\alpha}\xi^4\nabla\Delta z_1\cdot\nabla z_1 dx dt. \end{aligned}$$

Using the estimate

$$|\nabla(\rho_2^2 e^{-2s\alpha}\xi^4)| \le Cse^{-2s\alpha}\xi^5\rho_2, \quad s \ge C,$$

and again Young's inequality for the first term, we obtain

$$
\begin{aligned}
&s^2 \int_0^T \int_{\omega_3} \mathrm{e}^{-2s\alpha}\xi^4|\Delta z_1|^2 \mathrm{d}x\mathrm{d}t \\
\leq& C\Big(\underbrace{s^4 \int_0^T \int_{\omega_4} \mathrm{e}^{-2s\alpha}\xi^6|\nabla z_1|^2 \mathrm{d}x\mathrm{d}t}_{I_1} - \underbrace{s^2 \int_0^T \int_{\omega_4} \rho_2^2 \mathrm{e}^{-2s\alpha}\xi^4 \nabla\Delta z_1 \cdot \nabla z_1 \mathrm{d}x\mathrm{d}t}_{I_2}\Big)
\end{aligned}
\tag{2.65}
$$

for every $s \geq C$.
Now, to estimate I_1 we consider w_5 an open subset such that $w_4 \Subset w_5 \subset \omega'$ and $\rho_3 \in C_c^2(w_5)$ with $\rho_3 \equiv 1$ in ω_4 and $\rho_3 \geq 0$. Then

$$
\begin{aligned}
I_1 \leq& s^4 \int_0^T \int_{\omega_5} \rho_3 \mathrm{e}^{-2s\alpha}\xi^6|\nabla z_1|^2 \mathrm{d}x\mathrm{d}t \\
\leq& C\Big(s^6 \int_0^T \int_{\omega_5} \mathrm{e}^{-2s\alpha}\xi^8|z_1|^2 \mathrm{d}x\mathrm{d}t + s^4 \int_0^T \int_{\omega_5} \rho_3 \mathrm{e}^{-2s\alpha}\xi^6|\Delta z_1||z_1| \mathrm{d}x\mathrm{d}t\Big) \\
=& C\Big(s^6 \int_0^T \int_{\omega_5} \mathrm{e}^{-2s\alpha}\xi^8|z_1|^2 \mathrm{d}x\mathrm{d}t + s^4 \int_0^T \int_{\omega_5} \rho_3 \theta^*|\Delta z_1| \mathrm{e}^{-2s\alpha}(\theta^*)^{-1}\xi^6|z_1| \mathrm{d}x\mathrm{d}t\Big),
\end{aligned}
$$

for every $s \geq C$. We recall that $\theta^* := s^{1/2}\mathrm{e}^{-s\alpha^*}(\xi^*)^{9/22}$.
Using Young's inequality for the second term we obtain for every $\varepsilon > 0$:

$$
I_1 \leq \Big(s^6 \int_0^T \int_{\omega_5} \mathrm{e}^{-2s\alpha}\xi^8|z_1|^2 \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^T \int_{\omega_5} |\theta^*\Delta z_1|^2 \mathrm{d}x\mathrm{d}t + C(\varepsilon)s^7 \int_0^T \int_{\omega_5} \mathrm{e}^{-4s\alpha+2s\alpha^*}\xi^{12}|z_1|^2 \mathrm{d}x\mathrm{d}\Big),
\tag{2.66}
$$

for every $s \geq C$. The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (2.62).
Now we estimate I_2. An integration by parts gives

$$
I_2 \leq C\Big(s^3 \int_0^T \int_{\omega_4} \mathrm{e}^{-2s\alpha}\xi^5|\nabla\Delta z_1||z_1| \mathrm{d}x\mathrm{d}t + s^2 \int_0^T \int_{\omega_4} \mathrm{e}^{-2s\alpha}\xi^4|\Delta^2 z_1||z_1| \mathrm{d}x\mathrm{d}t\Big).
$$

Using again the Young's inequality, we obtain by an analogous argument the estimate:

$$
I_2 \leq C\Big(\varepsilon\|\hat{\theta} z_1\|^2_{L^2(0,T;H^4(\omega_4))} + C(\varepsilon)s^5 \int_0^T \int_{\omega_4} \mathrm{e}^{-4s\alpha+2s\alpha^*}\xi^{10}|z_1|^2 \mathrm{d}x\mathrm{d}t\Big),
\tag{2.67}
$$

for every $\varepsilon > 0$ and $s \geq C$. The first term in the right-hand side of (2.67) can be absorbed by the left-hand side of (2.62).

Finally, using the definition of the weight functions and (2.46), we readily obtain

$$s^7 \int_0^T \int_{\omega_5} \mathrm{e}^{-4s\alpha+2s\alpha^*} \xi^{12} |z_1|^2 \mathrm{d}x\mathrm{d}t$$

$$\leq 2s^7 \int_0^T \int_{\omega_5} \mathrm{e}^{-4s\hat{\alpha}+2s\alpha^*} (\hat{\xi})^{12} |\rho|^2 |\varphi_1|^2 \mathrm{d}x\mathrm{d}t + 2s^7 \int_0^T \int_{\omega_5} \mathrm{e}^{-4s\hat{\alpha}+2s\alpha^*} (\hat{\xi})^{12} |w_1|^2 \mathrm{d}x\mathrm{d}t$$

$$\leq 2s^7 \int_0^T \int_{\omega_5} \mathrm{e}^{-4s\hat{\alpha}+2s\alpha^*} (\hat{\xi})^{12} |\rho|^2 |\varphi_1|^2 \mathrm{d}x\mathrm{d}t + C\|\rho g\|^2_{L^2(Q)^2}.$$

From (2.62) and (2.63)-(2.67), we conclude the proof of Proposition 2.4.

2.4 Null controllability of the linear system

Here we are concerned with the null controllability of the following system:

$$\begin{cases} y_t - \nabla\cdot(Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (A(x,t)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{2.68}$$

where $y_0 \in W$, h is in an appropiate weighted space. We look for a control $v \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that $v_{\mathrm{i}} \equiv 0$ for some $\mathrm{i} \in \{1,\dots,N\}$.
To do this, let us first state a Carleman inequality with weight functions not vanishing in $t=0$.
Let $\ell \in C^2([0,T])$ be a positive function in $[0,T)$ such that $\ell(t) > t(T-t)$ for all $t \in [0,T/4]$ and $\ell(t) = t(T-t)$ for all $t \in [T/2,T]$.
Now, we introduce the following weight functions:

$$\begin{aligned} \beta(x,t) &= \frac{\mathrm{e}^{2\lambda\|\eta\|_\infty} - \mathrm{e}^{\lambda\eta(x)}}{\ell^{11}(t)}, & \gamma(x,t) &= \frac{\mathrm{e}^{\lambda\eta(x)}}{\ell^{11}(t)}, \\ \beta^*(t) &= \max_{x\in\overline{\Omega}} \beta(x,t), & \gamma^*(t) &= \min_{x\in\overline{\Omega}} \gamma(x,t), \\ \widehat{\beta}(t) &= \min_{x\in\overline{\Omega}} \beta(x,t), & \widehat{\gamma}(t) &= \max_{x\in\overline{\Omega}} \gamma(x,t). \end{aligned} \tag{2.69}$$

Lemma 2.8 *Let* $\mathrm{i} \in \{1,\dots,N\}$ *and let* s *and* λ *be like in Proposition 2.4. Then, there exists a constant* $C > 0$ *(depending on* s *and* λ *and increasing on* $\|A\|_{P^1_\varepsilon \cap P^2}$*) such that every solution* φ *of* (2.37) *satisfies:*

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N} + \iint_Q \mathrm{e}^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \mathrm{d}x\mathrm{d}t$$

$$\leq C\Big(\iint_Q \mathrm{e}^{-4s\beta^*} |g|^2 \mathrm{d}x\mathrm{d}t + \sum_{j=1, j\neq \mathrm{i}}^N \int_0^T \int_\omega \mathrm{e}^{-4s\hat{\beta}-2s\beta^*} (\hat{\gamma})^{12} |\chi_\omega \varphi_j|^2 \mathrm{d}x\mathrm{d}t \Big). \tag{2.70}$$

Proof. We start by an a priori estimate for the Stokes system (2.37). To do this, we introduce a function $\nu \in C^1([0,T])$ such that

$$\nu \equiv 1 \quad \text{in } [0,T/2], \qquad \nu \equiv 0 \quad \text{in } [3T/4,T].$$

We easily see that $(\nu\varphi, \nu\pi)$ satisfies

$$\left\{ \begin{array}{lll} -(\nu\varphi)_t - \nabla\cdot(D\nu\varphi) + \nabla(\nu\pi) = \nu g - \nu'\varphi & \text{in} & Q, \\ \nabla\cdot(\nu\varphi) = 0 & \text{in} & Q, \\ (\nu\varphi)\cdot n = 0,\ (\sigma(\nu\varphi,\nu\pi)\cdot n)_{tg} + (A^t(x,t)\nu\varphi)_{tg} = 0 & \text{on} & \Sigma, \\ (\nu\varphi)(T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{2.71}$$

Using (2.6) we have in particular

$$\begin{aligned} &\|\varphi\|_{L^2(0,T/2;L^2(\Omega)^N)} + \|\varphi(\cdot,0)\|_{L^2(\Omega)^N} \\ &\quad \le C\mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1+\|A\|^2_{P^0_\varepsilon}\Big)\Big(\|g\|_{L^2(0,3T/4;L^2(\Omega)^N)} + \|\varphi\|_{L^2(T/2,3T/4;L^2(\Omega)^N)}\Big). \end{aligned}$$

Taking into account that

$$\mathrm{e}^{-4s\beta^*} \ge C > 0 \ \ \forall t\in[0,3T/4] \quad \text{and} \quad \mathrm{e}^{-6s\beta^*}(\gamma^*)^3 \ge C > 0,\ \ \forall t \in [T/2,3T/4],$$

we have

$$\begin{aligned} &\int_0^{T/2}\!\!\int_\Omega \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t + \|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N} \\ &\quad \le C\mathrm{e}^{CT\|A\|^2_{P^0_\varepsilon}}\Big(1+\|A\|^2_{P^0_\varepsilon}\Big)\Big(\int_0^{3T/4}\!\!\int_\Omega \mathrm{e}^{-4s\beta^*}|g|^2\mathrm{d}x\mathrm{d}t + \int_{T/2}^{3T/4}\!\!\int_\Omega \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t\Big). \end{aligned} \tag{2.72}$$

Note that, since $\alpha = \beta$ in $\Omega\times(T/2,T)$, we have:

$$\int_{T/2}^{T}\!\!\int_\Omega \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t = \int_{T/2}^{T}\!\!\int_\Omega \mathrm{e}^{-6s\alpha^*}(\xi^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t$$

and by virtue of Carleman inequality (2.38) (see Proposition 2.4), we obtain

$$\int_{T/2}^{T}\!\!\int_\Omega \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t \le C\Big(\iint_Q \mathrm{e}^{-4s\alpha^*}|g|^2\mathrm{d}x\mathrm{d}t + \sum_{j=1,j\ne i}^{N}\int_0^T\!\!\int_{\omega'} \mathrm{e}^{-4s\hat\alpha-2s\alpha^*}(\hat\xi)^{12}|\varphi_j|^2\mathrm{d}x\mathrm{d}t\Big).$$

Since $\ell(t) = t(T-t)$ for any $t\in[T/2,T]$ and

$$\mathrm{e}^{-4s\beta^*} \ge C \quad \text{and} \quad \mathrm{e}^{-4s\hat\beta^*-2s\beta^*}(\hat\gamma)^{12} \ge C \ \ \text{in } [0,T/2],$$

we readily get

$$\int_{T/2}^{T}\!\!\int_\Omega \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t \le C\Big(\iint_Q \mathrm{e}^{-4s\beta^*}|g|^2\mathrm{d}x\mathrm{d}t + \sum_{j=1,j\ne i}^{N}\int_0^T\!\!\int_{\omega} \mathrm{e}^{-4s\hat\beta-2s\beta^*}(\hat\gamma)^{12}|\chi_\omega\varphi_j|^2\mathrm{d}x\mathrm{d}t\Big). \tag{2.73}$$

From (2.72) and (2.73) we obtain (2.70).

Remark 2.4 *Observe that on the left-hand side of* (2.70) *it is possible to add two terms and obtain*

$$\begin{aligned}&\|\mathrm{e}^{-3s\beta^*}(\gamma^*)^{9/22}\varphi\|^2_{L^2(0,T;H^2(\Omega)^N\cap W)}+\iint\limits_Q \mathrm{e}^{-6s\beta^*}(\gamma^*)^{9/11}|\varphi_t|^2\mathrm{d}x\mathrm{d}t\\&\quad+\|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N}+\iint\limits_Q \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\varphi|^2\mathrm{d}x\mathrm{d}t\\&\le C\Big(\iint\limits_Q \mathrm{e}^{-4s\beta^*}|g|^2\mathrm{d}x\mathrm{d}t+\sum_{j=1,j\neq \mathrm{i}}^{N}\int_0^T\int_\omega \mathrm{e}^{-4s\hat\beta-2s\beta^*}(\hat\gamma)^{12}|\chi_\omega\varphi_j|^2\mathrm{d}x\mathrm{d}t\Big).\end{aligned}\tag{2.74}$$

To this end, we consider $\tilde\theta:=\mathrm{e}^{-3s\beta^*}(\gamma^*)^{9/22}$ *and* $(\tilde\theta\varphi,\tilde\theta\pi)$ *the solution of* (2.71) *with* $\tilde\theta$ *instead of* ν. *Next, taking into account that* $|\partial_t\beta^*|\le C(\gamma^*)^{12/11}$, $|\tilde\theta'|\le C\mathrm{e}^{-3s\beta^*}(\gamma^*)^{3/2}$ *and the regularity estimate* (2.6)*, we obtain* (2.74).

Now we are ready to prove the null controllability of system (2.68). The idea is to look for a solution in an appropriate weighted functional space. Let us set

$$Ly=y_t-\nabla\cdot Dy$$

and let us introduce the space, for $N=2$ or $N=3$ and $\mathrm{i}\in\{1,\dots,N\}$,

$$\begin{aligned}E_N^{\mathrm{i}}:=\{&(y,p,v):\mathrm{e}^{2s\beta^*}y,\mathrm{e}^{2s\hat\beta+s\beta^*}(\hat\gamma)^{-6}v,\ \tilde\rho\partial_t v\in L^2(Q)^N,\ \tilde\rho v\in L^2(0,T;H^2(\Omega)^N),\\&v_{\mathrm{i}}\equiv 0,\mathit{supp}\,v\subset\omega\times(0,T),\ \mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y\in Y_1,\ \mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(Ly+\nabla p-v\chi_\omega)\in L^2(Q)^N\},\end{aligned}$$

where

$$\rho:=\mathrm{e}^{-4s\hat\beta-2s\beta*}(\hat\gamma)^{12}\quad\text{and}\quad\tilde\rho:=\rho^{-1}\tilde\theta.$$

It is clear that E_N^{i} is a Banach space for the following norm:

$$\begin{aligned}\|(y,p,v)\|_{E_N^{\mathrm{i}}}=\Big(&\|\mathrm{e}^{2s\beta^*}y\|^2_{L^2(Q)^N}+\|\mathrm{e}^{2s\hat\beta+s\beta^*}(\hat\gamma)^{-6}v\|^2_{L^2(Q)^N}+\|\tilde\rho\partial_t v\|^2_{L^2(Q)}\\&+\|\tilde\rho v\|^2_{L^2(0,T;H^2(\Omega)^N)}+\|\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y\|^2_{Y_1}\\&+\|\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(Ly+\nabla p-v\chi_\omega)\|^2_{L^2(Q)^N}\Big)^{1/2}.\end{aligned}$$

Remark 2.5 *Observe in particular that* $(y,p,v)\in E_N^{\mathrm{i}}$ *implies* $y(\cdot,T)=0$ *in* Ω.

Proposition 2.9 *Assume the hypothesis of Lemma 2.8 and*

$$y_0\in W\quad\textit{and}\quad \mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}h\in L^2(Q)^N.\tag{2.75}$$

Then, we can find a control v *such that the associated solution* (y,p) *to* (2.68) *satisfies* $(y,p,v)\in E_N^{\mathrm{i}}$. *In particular,* $v_{\mathrm{i}}\equiv 0$ *and* $y(\cdot,T)=0$ *in* Ω. *Furthermore, there exists* $C>0$ *increasing with respect to* $\|A\|_{P^1_\varepsilon\cap P^2}$ *such that*

$$\|v\|_{L^2(0,T;H^2(\omega)^N)}+\|v\|_{H^1(0,T;L^2(\omega)^N)}\le C\Big(\|y_0\|_{H^3(\Omega)^N\cap W}+\|h\|_{L^2(Q)^N}\Big).\tag{2.76}$$

Proof. Following the arguments in [FCGIP04], we introduce the space P_0 of functions $(\varphi, \pi) \in C^2(\overline{Q})^{N+1}$ such that

(i) $\nabla \cdot \varphi = 0 \quad \text{in } Q.$

(ii) $(\sigma(\varphi, \pi) \cdot n)_{tg} + (A^t(x,t)\varphi)_{tg} = 0 \quad \text{on } \Sigma.$

(iii) $\varphi \cdot n = 0 \quad \text{on } \Sigma.$

Also we define the bilinear form

$$a((\hat{\varphi}, \hat{\pi}), (w, q)) := \iint\limits_Q \mathrm{e}^{-4s\beta^*}(L^*\hat{\varphi} + \nabla\hat{\pi})(L^*w + \nabla q)\mathrm{d}x\mathrm{d}t$$

$$+ \sum_{j=1, j\neq \mathrm{i}}^{N} \int_0^T \int_\omega \mathrm{e}^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}\chi_\omega \hat{\varphi}_j \chi_\omega w_j \mathrm{d}x\mathrm{d}t,$$

for every $(w, q) \in P_0$, and a linear form

$$\langle G, (w, q)\rangle := \iint\limits_Q h \cdot w \,\mathrm{d}x\mathrm{d}t + \int\limits_\Omega y_0(\cdot) \cdot w(\cdot, 0)\,\mathrm{d}x, \tag{2.77}$$

where L^* is the adjoint operator of L, i.e.,

$$L^*w = -w_t - \nabla \cdot Dw.$$

Observe that Carleman inequality (2.70) holds for all $(w, q) \in P_0$. Consequently,

$$\iint\limits_Q \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|w|^2\mathrm{d}x\mathrm{d}t \le Ca((w, q), (w, q)), \quad \forall (w, q) \in P_0.$$

Therefore, $a(\cdot, \cdot) : P_0 \times P_0 \to \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot, \cdot)$. Then, $a(\cdot, \cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman inequality (2.70) and the assumption (4.16), the linear form $(w, q) \longmapsto \langle G, (w, q)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's Lemma, there exists one and only one $(\hat{\varphi}, \hat{\pi}) \in P$ satisfying:

$$a((\hat{\varphi}, \hat{\pi}), (w, q)) = \langle G, (w, q)\rangle, \quad \forall (w, q) \in P. \tag{2.78}$$

Let us set

$$\begin{cases} \hat{y} = \mathrm{e}^{-4s\beta^*}(L^*\hat{\varphi} + \nabla\hat{\pi}) \quad \text{in } Q, \\ \hat{v}_j = -\mathrm{e}^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}\hat{\varphi}_j\chi_\omega, \;\; j \neq \mathrm{i}, \;\; \hat{v}_\mathrm{i} \equiv 0 \text{ in } \omega \times (0, T). \end{cases} \tag{2.79}$$

Let us remark that $(\hat{y}, \hat{v})$ verifies

$$a((\hat{\varphi}, \hat{\pi}), (\hat{\varphi}, \hat{\pi})) = \iint\limits_Q \mathrm{e}^{-4s\beta^*}(L^*\hat{\varphi} + \nabla\hat{\pi})^2\mathrm{d}x\mathrm{d}t + \sum_{j=1, j\neq \mathrm{i}}^{N} \int_0^T \int_\omega \mathrm{e}^{-4s\hat{\beta}-2s\beta^*}(\hat{\gamma})^{12}|\chi_\omega\hat{\varphi}_j|^2\mathrm{d}x\mathrm{d}t$$

$$= \iint\limits_Q \mathrm{e}^{4s\beta^*}|\hat{y}|^2\mathrm{d}x\mathrm{d}t + \sum_{j=1, j\neq \mathrm{i}}^{N} \int_0^T \int_\omega \mathrm{e}^{4s\hat{\beta}+2s\beta^*}(\hat{\gamma})^{-12}|\hat{v}_j|^2\mathrm{d}x\mathrm{d}t < +\infty.$$

Let us prove that $\hat{y}$ is, together with some pressure $\hat{p}$, the weak solution of the Stokes system in (2.68) for $v = \hat{v}$. In fact, we introduce the (weak) solution $(\tilde{y}, \tilde{p})$ to the Stokes system:

$$\begin{cases} L\tilde{y} + \nabla\tilde{p} = h + \hat{v}\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot\tilde{y} = 0 & \text{in} \quad Q, \\ \tilde{y}\cdot n = 0,\ (\sigma(\tilde{y},\tilde{p})\cdot n)_{tg} + (A(x,t)\tilde{y})_{tg} = 0 & \text{on} \quad \Sigma, \\ \tilde{y}(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{2.80}$$

Clearly, $\tilde{y}$ is the unique solution of (2.80) defined by transposition. This means that $\tilde{y}$ is the unique function in $L^2(Q)^N$ satisfying

$$\iint\limits_Q \tilde{y}\cdot g \,\mathrm{d}x\mathrm{d}t = \int\limits_\Omega y_0(\cdot)\cdot w(\cdot,0)\mathrm{d}x + \iint\limits_Q h\cdot w \,\mathrm{d}x\mathrm{d}t + \iint\limits_Q \hat{v}\cdot w\chi_\omega \,\mathrm{d}x\mathrm{d}t, \quad \forall g\in L^2(Q)^N, \tag{2.81}$$

where w is, together with a pressure q, the solution to

$$\begin{cases} L^*w + \nabla q = g & \text{in} \quad Q, \\ \nabla\cdot w = 0 & \text{in} \quad Q, \\ w\cdot n = 0, (\sigma(w,q)\cdot n)_{tg} + (A^t(x,t)w)_{tg} = 0 & \text{on} \quad \Sigma, \\ w(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

From (2.78) and (2.79), we see that $\hat{y}$ also satisfies (2.81). Consequently, $\hat{y} = \tilde{y}$ and $\hat{y}$ is, together with $\hat{p} = \tilde{p}$, the weak solution to the Stokes system (2.80).
Finally, we must see that $(\hat{y},\hat{p},\hat{v}) \in E_N^{\mathrm{i}}$. We already know that

$$\mathrm{e}^{2s\beta^*}\hat{y},\ \mathrm{e}^{2s\hat{\beta}+s\beta^*}(\hat{\gamma})^{-6}\hat{v} \in L^2(Q)^N$$

and (see (4.16))

$$\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(L\hat{y} + \nabla\hat{p} - \hat{v}\chi_\omega) \in L^2(Q)^N.$$

Thus, it only remains to check that

$$\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}\hat{y} \in Y_1 \quad \text{and} \quad \tilde{\rho}\hat{v} \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N).$$

i) We define the functions

$$y^* := \mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}\hat{y}, \quad p^* := \mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}\hat{p}$$

and

$$h^* := \mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}(h + \hat{v}\chi_\omega).$$

Then (y^*, p^*) satisfies:

$$\begin{cases} Ly^* + \nabla p^* = h^* + (\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11})'\hat{y} & \text{in} \quad Q, \\ \nabla\cdot y^* = 0 & \text{in} \quad Q, \\ y^*\cdot n = 0,\ (\sigma(y^*,p^*)\cdot n)_{tg} + (A(x,t)y^*)_{tg} = 0 & \text{on} \quad \Sigma, \\ y^*(\cdot,0) = \mathrm{e}^{2s\beta^*(0)}(\gamma^*(0))^{-12/11}y_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

Since $h^* + (\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11})'\hat{y} \in L^2(Q)^N$ and $y_0 \in W$, we have $y^* \in Y_1$ (see Lemma 2.2 in Section 2).

ii) Now, let us bound the $H^1(0,T;L^2(\omega)^N)$ and the $L^2(0,T;H^2(\omega)^N)$ norms of the control. Using (2.79), we obtain

$$\sum_{j=1,j\neq \mathrm{i}}^{N} \int_0^T \tilde{\rho}^2(\|\partial_t \hat{v}_j\|^2_{L^2(\omega)} + \|\hat{v}_j\|^2_{H^2(\omega)})\mathrm{d}x\mathrm{d}t$$
$$\leq C \sum_{j=1,j\neq \mathrm{i}}^{N} \Big(\iint_Q \mathrm{e}^{-6s\beta^*}(\gamma^*)^3|\hat{\varphi}_j|^2\mathrm{d}x\mathrm{d}t + \iint_Q \tilde{\theta}^2|\partial_t\hat{\varphi}_j|^2\mathrm{d}x\mathrm{d}t + \|\tilde{\theta}\hat{\varphi}_j\|^2_{L^2(0,T;H^2(\Omega))}\Big).$$

Taking into account that (2.70) and Remark 2.4 hold for all $(\hat{\varphi},\hat{\pi}) \in P_0$, we readily obtain

$$\sum_{j=1,j\neq \mathrm{i}}^{N} \int_0^T \tilde{\rho}^2(\|\partial_t \hat{v}_j\|^2_{L^2(\omega)} + \|\hat{v}_j\|^2_{H^2(\omega)})\mathrm{d}x\mathrm{d}t \leq Ca((\hat{\varphi},\hat{\pi}),(\hat{\varphi},\hat{\pi})). \tag{2.82}$$

Finally, from the continuity of G (see (2.77)) and (2.78), we deduce (2.76). This ends the proof of Proposition 2.9.

2.5 Proof of the main result

In this section we give the proof of Theorem 2.1 using classical arguments. The first step is to apply Kakutani's fixed point theorem on the boundary. Finally, we will deal with the nonlinear term in the Navier-Stokes equations through an inverse mapping theorem to conclude the proof of Theorem 2.1.

2.5.1 Nonlinearity on the boundary conditions.

In this section we present the local null controllability for the following system:

$$\begin{cases} y_t - \nabla\cdot(Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ y\cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (f(y))_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{2.83}$$

Theorem 2.10 *Let us assume that $f \in C^4(\mathbb{R}^N;\mathbb{R}^N)$ and $f(0)=0$. Then, for every $T>0$, $\omega\subset\Omega$ and $\mathrm{i}\in\{1,\dots,N\}$, there exists $\delta>0$ such that, for every $y_0\in H^3(\Omega)^N\cap W$, $h\in Y_1$ satisfying $\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}h\in L^2(Q)^N$,*

$$\|h\|_{Y_1} + \|y_0\|_{H^3(\Omega)^N\cap W} \leq \delta \tag{2.84}$$

and the compatibility condition (2.2), we can find a control

$$v\in L^2(0,T;H^2(\omega)^N)\cap H^1(0,T;L^2(\omega)^N)$$

and an associated solution (y,p) of (2.83) satisfying $y\in Y_2$ and such that $(y,p,v)\in E_N^{\mathrm{i}}$.

Proof. For every $z \in Z_\varepsilon$ (recall that Z_ε was defined in (2.4)) we consider the following system:

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y \cdot n = 0,\ (\sigma(y,p)\cdot n)_{tg} + (g(z)y)_{tg} = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{2.85}$$

where

$$g(z) := \frac{1}{N}\int_0^1 \nabla f(\tau z)\mathrm{d}\tau.$$

On the other hand, observe that since $f \in C^4(\mathbb{R}^N;\mathbb{R}^N)$, each row and each column of $g(z)$ belongs to Z_ε. Then, for every $z \in Z_\varepsilon$ we can use Proposition 2.9 with $A = g(z)$ and deduce the existence of a control v_z belonging to $L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that the solution (y_z, p_z) of (2.85) satisfies $(y_z, p_z, v_z) \in E_N^{\mathrm{i}}$.
Moreover, from (2.76) we have

$$\|v_z\|_{L^2(0,T;H^2(\omega)^N)} + \|v_z\|_{H^1(0,T;L^2(\omega)^N)} \le C_1(\Omega, \omega, T, \|g(z)\|_{P_\varepsilon^1 \cap P^2})\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{L^2(Q)^N}\Big), \tag{2.86}$$

where C_1 is increasing with respect to $\|g(z)\|_{P_\varepsilon^1\cap P^2}$.

Next, taking into account that $v_z, h \in Y_1$ and the compatibility condition (2.7) with u_0 replaced by y_0, $A(\cdot,0)$ replaced by $g(y_0(\cdot))$ and $f_2(\cdot,0)$ replaced by 0 (see (2.2)), we can apply Theorem 2.3 to system (2.85). Combining this with (2.86), we can obtain that $y_z \in Y_2$ and

$$\|y_z\|_{Y_2} \le C_2(\Omega, \omega, T, \|g(z)\|_{P_\varepsilon^1\cap P^2})\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{Y_1}\Big), \tag{2.87}$$

with C_2 increasing with respect to $\|g(z)\|_{P_\varepsilon^1\cap P^2}$ (see (2.10)).

Let $\mathcal{C}(z)$ be the set constituted by the controls $v_z \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ that satisfy (2.86) and drive the solution y_z of system (2.85) to zero at time T. Then, let us introduce

$$\Lambda(z) := \{y_z \text{ solution of (2.85) } : v_z \in \mathcal{C}(z)\}.$$

Observe that, thanks to (2.87), $\Lambda(z)$ is included in Y_2. Moreover, for any $z \in Y_2$ such that $\|z\|_{Y_2} \le 1$, we have $\|g(z)\|_{P_\varepsilon^1\cap P^2} \le M$, where $M > 0$ is a constant only depending on ε, T and Ω. Consequently,

$$\|y_z\|_{Y_2} \le C_2(\Omega, \omega, T, M)\Big(\|y_0\|_{H^3(\Omega)^N\cap W} + \|h\|_{Y_1}\Big)$$

(see (2.87)). Choosing now $\delta := \dfrac{1}{C_2(\Omega,\omega,T,M)}$ in (2.84), we find $\|y_z\|_{Y_2} \le 1$.

Now, we want to establish that the set-valued map $\Lambda : K \to 2^K$ possesses a fixed-point, where

$$K := \overline{B}_{Y_2}(0;1) = \{y \in Y_2 : \|y\|_{Y_2} \le 1\}.$$

For this end, we will apply Kakutani's fixed-point theorem (see for instance [AF09], Theorem 3.2.3, page 87):

i) $\Lambda(z)$ is a nonempty closed convex set of $L^2(Q)^N$, for every $z \in K$.
ii) K is a nonempty convex compact set of $L^2(Q)^N$.
iii) Λ is upper-hemicontinuous in $L^2(Q)^N$, i.e, for any $\lambda \in L^2(Q)^N$, the mapping

$$z \to \sup_{y \in \Lambda(z)} \langle \lambda, y \rangle_{L^2(Q)^N}$$

is upper semicontinuous.

i) For every $z \in K$, let $(y_z^k) \subset \mathcal{C}(z)$ such that $y_z^k \to y_z$ in $L^2(Q)^N$. From (2.86), we find (at least for a subsequence) that $v_z^{k'} \to v_z$ in $L^2(Q)^N$. Let us denote w_z the solution of (2.85) associated to $v := v_z$. Then, $y_z^{k'} - w_z$ satisfies (2.85) with $h := 0$, $v := v_z^{k'} - v_z$ and $y_0 := 0$. Thanks to (2.6), we have $y_z^{k'} \to w_z$ in $L^2(Q)^N$ in particular and so $y_z = w_z$. This shows that $\Lambda(z)$ is closed. The convexity of $\Lambda(z)$ is trivial.
ii) Since Y_2 is compactly embedeed into $L^2(Q)^N$, the second item holds true.
iii) Finally, let us prove the upper-hemicontinuity of Λ. Assume $z_k \to z$ in $L^2(Q)^N$. In consequence from the compactness of $\Lambda(z_k)$, we have

$$\sup_{y \in \Lambda(z_k)} \langle \lambda, y \rangle_{L^2(Q)^N} = \langle \lambda, y_k \rangle_{L^2(Q)^N},$$

for some $y_k \in \Lambda(z_k)$. Then, we choose $(z_{k'}) \subset (z_k)$ such that

$$\lim_{k' \to \infty} \sup_{y \in \Lambda(z_{k'})} \langle \lambda, y \rangle_{L^2(Q)^N} = \lim_{k' \to \infty} \langle \lambda, y_{k'} \rangle_{L^2(Q)^N}$$

and denote $v_{k'}$ the controls in $\mathcal{C}(z_{k'})$ which are associated to $y_{k'} \in \Lambda(z_{k'})$. From (2.86), there exists $v^* \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that $v_{k'} \rightharpoonup v^*$ in $L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ and $v^* \in \mathcal{C}(z)$. In particular, $v_{k'} \to v^*$ in $L^2(Q)^N$ (for a subsequence). Now, let (y^*, p^*) be the solution to (2.85) associated to v^*. We set $\tilde{y}_{k'} := y_{k'} - y^*$, $\tilde{p}_{k'} := p_{k'} - p^*$ and $\tilde{v}_{k'} := v_{k'} - v^*$. Then,

$$\begin{cases} (\tilde{y}_{k'})_t - \nabla \cdot (D\tilde{y}_{k'}) + \nabla \tilde{p}_{k'} = \tilde{v}_{k'} \chi_\omega & \text{in} \quad Q, \\ \nabla \cdot \tilde{y}_{k'} = 0 & \text{in} \quad Q, \\ \tilde{y}_{k'} \cdot n = 0, \ (\sigma(\tilde{y}_{k'}, \tilde{p}_{k'}) \cdot n)_{tg} + (g(z)\tilde{y}_{k'})_{tg} = ([g(z) - g(z_{k'})]y_{k'})_{tg} & \text{on} \quad \Sigma, \\ \tilde{y}_{k'}(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases}$$

Taking into account that $g(z_{k'}) \to g(z)$ in Z_ε, one can prove that in particular

$$\|[g(z) - g(z_{k'})]y_{k'}\|_{L^2(0,T;H^{1/2}(\partial\Omega)^N) \cap H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial\Omega)^N)} \xrightarrow{k' \to \infty} 0.$$

Then, from Lemma 2.2 we can deduce that $y_{k'} \to y^*$ in Y_1. Additionally, $y^* \in \Lambda(z)$ and therefore,

$$\lim_{k' \to \infty} \sup_{y \in \Lambda(z_{k'})} \langle \lambda, y \rangle_{L^2(Q)^N} = \lim_{k' \to \infty} \langle \lambda, y_{k'} \rangle_{L^2(Q)^N} = \langle \lambda, y^* \rangle_{L^2(Q)^N} \leq \sup_{y \in \Lambda(z)} \langle \lambda, y \rangle.$$

This concludes the proof of Theorem 2.10.

2.5.2 Nonlinearity in the main equation.

Theorem 2.11 *Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces and*

$$\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$$

is a continuously differentiable map. We assume that for $b_1^0 \in \mathcal{B}_1, b_2^0 \in \mathcal{B}_2$ the equality

$$\mathcal{A}(b_1^0) = b_2^0 \tag{2.88}$$

holds and $\mathcal{A}'(b_1^0) : \mathcal{B}_1 \to \mathcal{B}_2$ is an epimorphism. Then there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ which satisfies the condition

$$\|b_2^0 - b_2\|_{\mathcal{B}_2} < \delta$$

there exists a solution $b_1 \in \mathcal{B}_1$ of the equation

$$\mathcal{A}(b_1) = b_2.$$

We apply this theorem for some given $\mathrm{i} \in \{1, \dots, N\}$ and the spaces

$$\mathcal{B}_1 := \{(y, p, v) \in E_N^{\mathrm{i}} : y \in Y_2\}$$

and

$$\mathcal{B}_2 := \{(h, y_0) \in [L^2(\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(0,T); L^2(\Omega)^N) \cap Y_1] \times [H^3(\Omega)^N \cap W] : h, y_0 \text{ satisfies (2.84)}\}$$

We define the operator $\mathcal{A}$ by the formula

$$\mathcal{A}(y, p, v) = (Ly + (y \cdot \nabla)y + \nabla p - v\chi_\omega, y(\cdot, 0)).$$

Let us see that $\mathcal{A}$ is of class $C^1(\mathcal{B}_1, \mathcal{B}_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(y \cdot \nabla)y$. We prove now that the bilinear operator

$$((y^1, p^1, v^1), (y^2, p^2, v^2)) \longmapsto (y^1 \cdot \nabla)y^2$$

is continuous from $\mathcal{B}_1 \times \mathcal{B}_1$ to $L^2(\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(0,T); L^2(\Omega)^N) \cap Y_1$.
In fact, notice that (see the definition of the space E_N^{i}):

$$\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11} y \in L^2(0,T; L^\infty(\Omega)^N)$$

and

$$\nabla(\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11} y) \in L^\infty(0,T; L^2(\Omega)^{N\times N}).$$

Consequently, we obtain

$$\begin{aligned}
&\|\mathrm{e}^{3s\beta^*}(\gamma^*)^{-3/2}(y^1 \cdot \nabla)y^2\|_{L^2(Q)^N} \\
&\leq C\|(\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y^1 \cdot \nabla)\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y^2\|_{L^2(Q)^N} \\
&\leq C\|\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|\mathrm{e}^{2s\beta^*}(\gamma^*)^{-12/11}y^2\|_{L^\infty(0,T;W)}.
\end{aligned}$$

On the other hand,

$$\|(y^1 \cdot \nabla)y^2\|_{Y_1} \leq C\|y^1\|_{Y_2}\|y^2\|_{Y_2}.$$

Notice that $\mathcal{A}'(0,0,0) : \mathcal{B}_1 \to \mathcal{B}_2$ is given by

$$\mathcal{A}'(0,0,0)(y,p,v) = (Ly + \nabla p - v\chi_\omega, y(\cdot,0)), \quad \text{for all } (y,p,v) \in \mathcal{B}_1.$$

In virtue of Theorem 2.10, this functional satisfies $Im(\mathcal{A}'(0,0,0)) = \mathcal{B}_2$.
Let $b_1^0 = (0,0,0)$ and $b_2^0 = (0,0)$. Then equation (2.88) obviously holds. So all necessary conditions to apply Theorem 2.11 are fulfilled. Therefore there exists a positive number δ such that, if $\|y(\cdot,0)\|_{H^3(\Omega)^N \cap W} \leq \delta$, we can find a control v satisfying $v_{\mathrm{i}} \equiv 0$, for some given $\mathrm{i} \in \{1,\ldots,N\}$ and an associated solution (y,p) to (2.1) satisfying $y(\cdot,T) = 0$ in Ω. This finishes the proof of Theorem 2.1.

Chapter 3

First inverse source problem for the Stokes system

3.1 Introduction

We consider the inverse problem of determining the spatial dependence of a source in the Stokes system of the form $f(x)\sigma(t)$ defined in $\Omega \times (0,T)$, assuming that $\sigma(t)$ is known and $f(x)$ is divergence free. The only available observations are single internal measurements of the velocity, in which one of its components is missing. Under some hypothesis on σ we prove uniqueness of this inverse problem via some explicit reconstruction formula. This formula provides the spectral coefficients f_k of the source f in terms of a family of null controls $h^{(\tau)}$ for the corresponding dual system indexed by $\tau \in (0,T]$. Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) with smooth boundary Γ. Let $T > 0$ and let $\omega \subset \Omega$ be an arbitrary nonempty subdomain. Given an initial data y_0, we consider the following Stokes system:

$$\begin{cases} y_t - \nu\Delta y + \nabla p = F(x,t) & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot y = 0 & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \Gamma \times (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where $F(x,t) = f(x)\sigma(t)$ represents the source term or density of external forces causing the movement of the fluid and $\nu > 0$ is the diffusion coefficient. Let us now introduce usual spaces in the context of problems modeling incompressible fluids:

$$V := \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H := \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, \;\; y \cdot n = 0 \text{ on } \Gamma\},$$

where $n(x)$ is the outward unit normal vector to Ω at the point $x \in \Gamma$.

It is well known that if $F \in L^2(0,T;H)$ and $y_0 \in V$, then there exists a unique solution (y,p) for the system (3.1) such that $y \in L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))^N)\cap H^1(0,T;L^2(\Omega)^N)$ and $p \in H^1(0,T;L^2(\Omega))$.

Our aim is to establish a reconstruction formula for the following inverse problem: determining the source $f(x)$ in the system (3.1) from local and missing velocity data. That is to say, from $N-1$ scalar components of the velocity field y and its time derivative y_t in some strict subset or observatory $\omega \subset \Omega$ measured during a time interval $(0,T)$.

Inverse problems of this type for the Stokes or Navier-Stokes system have been not studied intensively. The closest-related results can be found in [CIPY13], [IY00] and [Mar15]. In [CIPY13], the authors proved the Lipschitz stability of recovering the spatially part of a source term for the linearized Navier-Stokes equations with data $y|_{\omega_1\times(0,T)}, y|_{\{\theta\}\times\Omega}$ where $\omega_1 \subset \Omega$ is an arbitrary subdomain and $0<\theta<T$. In this case, the density of external force is $F = R(x,t)g(x)$, where $R(x,t)$ is a vector-valued function known and $g(x)$ is unknown. On the other hand, in [IY00] the authors considered the same external force as in this work $F = f(x)\sigma(t)$, but they focus on recovering f from data $y|_{\omega_2\times(0,T)}$, $p|_{\omega_2\times(0,T)}$, $y|_{\{\theta\}\times\Omega}$, $p|_{\{\theta\}\times\Omega}$, where ω_2 is an arbitrary subdomain and $0<\theta<T$. In all these studies, the arguments are based on the general Bukhgeim-Klibanov method to obtain stability based on global Carleman estimates [Kli81]. We also refer to the more recent work [Mar15], where the authors use spectral analysis on unsteady Stokes/Brinkman system in order to prove identification results for (F,g), where F is the external source and $g = \nabla \cdot y$ is the compressible source term. In this article, the identification is obtained from one or several spectral measurement of the normal component of the stress tensor on the whole boundary.

There exists a complete different approach to this problem based on the relationship between null-controllability and inverse problems. This method was firstly developed for hyperbolic equations in [Yam95] and then extended to parabolic equations in [GOT13]. The advantage of this methods is that they provide an explicit recovery formula for the source $f(x)$ in terms of local measurements and null-controls. The main difference between the hyperbolic and the parabolic case is that in the first case just one type of null-controls are required (controlling from T to 0) meanwhile, in the second case a family of null-controls appears (controlling from τ to 0 for $\tau \in (0,T]$.

Also using the connection between null controllability and several inverse problems, in [GT11], the authors study the conditional logarithmic stability for the source inverse problem for a wide class of parabolic equations for regular enough sources and from internal or boundary measurements. The results are then extended to the Stokes system.

Our main results, Theorem 3.4 and Theorem 3.5, provide a reconstruction formula of each Fourier coefficient of f by means of $N-1$ components of local measurements of the solution y of system (3.1). The main ideas for obtaining this formula have been taken from [GOT13]. However, the full adaptation to the Stokes system (3.1) has the following new challenges:

- We will able to recover only the divergence-free part of the source f from the local (in space) velocity, but without measuring the pressure. This makes a difference with the previous works [IY00] and [CIPY13].
- Instead of using the classical null-controllability results for the Stokes system (see for instance [FI96b], [FCGBGP06]), we have to consider [CG09], where the authors obtain the null-controllability for the N-dimensional Stokes system with $N-1$ scalar controls through Carleman inequalities. This fact allow us, by duality, to consider local measurements of the velocity with one missing component for the reconstruction. Under our knowledge, this is a completely new application of the global Carleman inequalities with missing components of this type.
- Numerically, in order to approximate a null-control with one vanishing component, it is necessary to introduce two regularizing parameters $\alpha > 0$ and $\beta > 0$. The first one is classical (see for instance [GLH08], [Lio71]) and it serves to penalize the exact null final condition. The other parameter is new and it is added in order to penalize the vanishing component. This generalize the case considered in [GOT13] to missing components in the multidimensional case.

This chapter is organized as follows. In Section 3.2 we first prove the uniqueness and reconstruction results, Theorem 3.4 and Theorem 3.5. Next, in Section 3.3 we give a method to approximate null controls with one vanishing component and prove its convergence. Finally, in Section 3.4 we implement this method and present several numerical experiments that show the feasibility of the proposed recovering formula.

Before starting with Section 3.2, we recall some preliminary lemmas concerning the null controllability of Stokes system using null controls with one vanishing component.
The following Theorem was proved in [CG09] and establishes the null controllability for the N-dimensional Stokes system with one vanishing in the control using Carleman inequalities.

Lemma 3.1 *Given $\tau \in (0,T]$, $\omega \subset \Omega$ with nonempty interior and $\varphi_0 \in H$, there exists a control $h^{(\tau)} = h^{(\tau)}(\varphi_0) \in L^2(0,\tau; L^2(\Omega)^N)$ with $h_j^{(\tau)} \equiv 0$ for some $j \in \{1, \cdots, N\}$, such that the solution ϕ of the problem*

$$\begin{cases} -\phi_t - \nu\Delta\phi + \nabla\pi = h^{(\tau)} 1_{\omega\times[0,\tau]} & \text{in} \quad \Omega\times(0,\tau), \\ \nabla\cdot\phi = 0 & \text{in} \quad \Omega\times(0,\tau), \\ \phi = 0 & \text{on} \quad \Gamma\times(0,\tau), \\ \phi(\cdot,\tau) = \varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{3.2}$$

satisfies

$$\phi(\cdot,0) = 0 \quad in \quad \Omega. \tag{3.3}$$

Moreover, there exist constants $C_0 > 0$ and $C_1 > 0$ depending only on Ω and ω such that

$$\|h^{(\tau)}\|_{L^2(0,T;L^2(\omega)^N)} \le C_0 \mathrm{e}^{C_1/\tau^9} \|\varphi_0\|_{L^2(\Omega)^N}. \tag{3.4}$$

Remark 3.1 *The proof of Lemma 3.1 is equivalent to the following observability inequality:*

$$\|w(\tau)\|^2_{L(\Omega)} \leq C_0 \mathrm{e}^{C_1/\tau^9} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^\tau \int_\omega |w_\mathrm{i}|^2 \mathrm{d}x\mathrm{d}t, \tag{3.5}$$

where (w, q) *is the solution of the adjoint system*

$$\begin{cases} w_t - \nu\Delta w + \nabla q = 0 & \text{in} \quad \Omega \times (0,\tau), \\ \nabla \cdot w = 0 & \text{in} \quad \Omega \times (0,\tau), \\ w = 0 & \text{on} \quad \Gamma \times (0,\tau), \\ w(\cdot, 0) \ \textit{given} & \text{in} \quad \Omega. \end{cases} \tag{3.6}$$

Finally, we recall technical results about the Volterra equations of first and second kind we need afterwards. For more details, the interested reader can see [GOT13], [Tri57] and [Yam95].

Lemma 3.2 *For* $0 < t < \tau < T$ *and every* $\eta \in L^2(0,\tau; L^2(\Omega)^N)$*, there exists a unique*

$$\theta \in H^1(0,\tau; L^2(\Omega)^N)$$

satisfying for every $\mathrm{i} \in \{1,\ldots,N\}$ *the Volterra equation of the second kind*

$$\begin{aligned} &\sigma(0)\partial_t\theta_\mathrm{i}(x,t) + \int_t^\tau (\sigma(s-t)\theta_\mathrm{i}(x,s) + \sigma'(s-t)\partial_t\theta_\mathrm{i}(x,s))\mathrm{d}s = \eta_\mathrm{i}(x,t), \\ &\theta_\mathrm{i}(x,\tau) = 0. \end{aligned} \tag{3.7}$$

Furthermore, there exists a constant $C > 0$ *depending on* $\|\sigma\|_{W^{1,\infty}(0,\tau)}$ *such that*

$$\|\theta\|_{H^1(0,\tau;L^2(\Omega)^N)} \leq C\|\eta\|_{L^2(0,\tau;L^2(\Omega)^N)}. \tag{3.8}$$

Lemma 3.3 *We define the operators* $K : L^2(0,T;L^2(\Omega)^N) \to H^1(0,T;L^2(\Omega)^N)$ *and* $L : L^2(0,T;H^1(\Omega)) \to L^2(0,T;H^1(\Omega))$ *by*

$$(Kv)(x,t) := \int_0^t \sigma(s)v(x,t-s)\mathrm{d}s, \quad (Lq)(x,t) := \int_0^t \sigma(s)q(x,t-s)\mathrm{d}s. \tag{3.9}$$

There exists a positive constant C *depending only on* Ω, T *and* $\|\sigma\|_{W^{1,\infty}(0,T)}$ *such that*

$$C\|Kv\|_{H^1(0,T;L^2(\Omega)^N)} \leq \|v\|_{L^2(Q)^N} \leq \|Kv\|_{H^1(0,T;L^2(\Omega)^N)}. \tag{3.10}$$

$$C\|Lq\|_{L^2(0,T;H^1(\Omega))} \leq \|q\|_{L^2(Q)^N} \leq \|Lq\|_{L^2(0,T;H^1(\Omega)^N)}.$$

Furthermore, the adjoint operator $K^* : H^1(0,T;L^2(\Omega)^N) \to L^2(0,T;L^2(\Omega)^N)$ *is given by*

$$(K^*\theta)(x,t) = \sigma(0)\partial_t\theta(x,t) + \int_t^T (\sigma(s-t)\theta(x,s) + \sigma'(s-t)\partial_t\theta(x,t))\mathrm{d}s. \tag{3.11}$$

3.2 Uniqueness and reconstruction with one missing component

We now address the uniqueness and the reconstruction of the inverse source problem for the Stokes system (3.1) following the same ideas of [GOT13]. The main differences here are on one side that we are in presence of a systems of N equations and we have to project into the H space in order to eliminate the pressure. On the other side, we observe the velocity with one missing component, so we should use by duality null-controls with one vanishing component.

Our first result is given in the following theorem (analogous to Theorem 1.3 in [GOT13]).

Theorem 3.4 *Let $\sigma \in W^{1,\infty}(0,T)$ with $\sigma(T) \neq 0$. Given $\varphi_0 \in H$, for each $0 < \tau \leq T$, let $h^{(\tau)} = (h_j^{(\tau)})_{j=1}^N$ be a null control associated to problem (3.2) extended by zero in $(\tau, T]$ with $h_j^{(\tau)} \equiv 0$ for some $j \in \{1, \cdots, N\}$. Let $\theta^{(\tau)}$ be a solution of (3.7) for $\eta = h^{(\tau)}$ extended by zero in $(\tau, T]$. Then*

$$(f, \varphi_0)_{L^2(\Omega)^N} = \mathcal{L} + \mathcal{C}_1 + \mathcal{C}_2,$$

where

$$\begin{aligned}
\mathcal{L}(\varphi_0) &= -\frac{\nu}{\sigma(T)}(\Delta y(\cdot, T), \varphi_0)_{L^2(\Omega)^N},\\
\mathcal{C}_1 &= -\frac{\sigma(0)}{\sigma(T)} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (y_\mathrm{i}, \theta_\mathrm{i}^{(T)})_{H^1(0,T;L^2(\omega))},\\
\mathcal{C}_2 &= -\frac{1}{\sigma(T)} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^T \sigma'(T-s)(y_\mathrm{i}, \theta_\mathrm{i}^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.
\end{aligned} \tag{3.12}$$

Moreover, if $\sigma'(t) = 0$ for $t \in (T-\varepsilon, T]$ for some $\varepsilon > 0$ or $\sigma'(t) = \mathrm{e}^{-C/(T-t)^9}\rho(t)$ for all $t \in (0,T)$, $\rho \in L^\infty(0,T)$ for large C, then we obtain the stability inequality

$$\|f\|_{L^2(\Omega)^N} \leq C\Big(\|\Delta y(\cdot, T)\|_{L^2(\Omega)^N} + \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \|y_\mathrm{i}\|_{H^1(0,T;L^2(\omega))}\Big) \tag{3.13}$$

with $C \sim O(\mathrm{e}^{C_1/\varepsilon^9})$ and C_1 is the constant appearing in (3.4).

Remark 3.2 *Notice that the reconstruction formula (3.12) involves a system of equations and one missing component of the velocity in the observatory $\omega \times (0,T)$ since we consider a family of exact controls $h^{(\tau)}$ having one vanishing component. This is the main difference with the reconstruction formula presented in [GOT13] for scalar parabolic equations.*

Proof of Theorem 3.4. Using the operators K and L defined in Lemma 3.3 it is easy to see that if (w, q) satisfies (3.6) with initial condition $w(0) = f$ then $y = Kw$ and $p = Lq$ satisfy (3.1). Evaluating the main equation (3.1) in T, using that $y_t(T) = \sigma(0)w(T) + \int_0^T \sigma'(T-s)w(x,s)\mathrm{d}s$, after multiplying by $\varphi_0 \in H$ and integrating in space, we easily deduce that

$$\begin{aligned}\sigma(T)(f,\varphi_0)_{L^2(\Omega)^N} = \;& \sigma(0)(w(\cdot,T),\varphi_0)_{L^2(\Omega)^N} - \nu(\Delta y(\cdot,T),\varphi_0)_{L^2(\Omega)^N} \\ & + \int_0^T \sigma'(T-s)(w(\cdot,s),\varphi_0)_{L^2(\Omega)^N}\mathrm{d}s\end{aligned} \tag{3.14}$$

since

$$(\nabla p(\cdot,T),\varphi_0)_{L^2(\Omega)^N} = 0.$$

Next, observe that for all $\tau \in (0,T]$, the term $(w(\cdot,\tau),\varphi_0)_{L^2(\Omega)^N}$ can be evaluated by multiplying the principal equation in (3.6) by ϕ solution of the control system (3.2), and after using integration by parts in the domain $\Omega \times (0,\tau)$. Then, if $h^{(\tau)}$ is extended by zero for $\tau < t < T$ we have

$$(w(\cdot,\tau),\varphi_0)_{L^2(\Omega)^N} = -\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^T \int_\omega w_{\mathrm{i}}(x,t) h_{\mathrm{i}}^{(\tau)}(x,t)\mathrm{d}x\mathrm{d}t. \tag{3.15}$$

On the other hand, from (3.7) and (3.11) we can consider the Volterra equations: $K^*(\theta_{\mathrm{i}}^{(\tau)}) = h_{\mathrm{i}}^{(\tau)}, \quad \mathrm{i} \in \{1,\ldots,N\}, \mathrm{i} \neq j$, where $\theta_{\mathrm{i}}^{(\tau)}(t) = 0$ for $\tau \le t \le T$. Then, by solving these problems and using $y = Kw$ we obtain

$$(w(\cdot,\tau),\varphi_0)_{L^2(\Omega)^N} = -\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (w_{\mathrm{i}}, K^*\theta_{\mathrm{i}}^{(\tau)})_{L^2(0,T;L^2(\omega))} = -\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (y_{\mathrm{i}}, \theta_{\mathrm{i}}^{(\tau)})_{H^1(0,T;L^2(\omega))}.$$

Hence, applying the above identity in (3.14) for every $\varphi_0 \in H$, we have

$$\begin{aligned}(f,\varphi_0)_{L^2(\Omega)^N} = \;& -\frac{\sigma(0)}{\sigma(T)} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (y_{\mathrm{i}}, \theta_{\mathrm{i}}^{(T)})_{H^1(0,T;L^2(\omega))} - \frac{\nu}{\sigma(T)}(\Delta y(\cdot,T),\varphi_0)_{L^2(\Omega)^N} \\ & -\frac{1}{\sigma(T)} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^T \sigma'(T-s)(y_{\mathrm{i}}, \theta_{\mathrm{i}}^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.\end{aligned} \tag{3.16}$$

The stability result (3.13) is deduced following the same proof as in [GOT13] Theorem 1.3, from (3.4) and (3.8) since

$$\|\theta^{(\tau)}\|_{H^1(0,\tau;L^2(\Omega)^N)} \le C\|h^{(\tau)}\|_{L^2(0,\tau;L^2(\Omega)^N)} \le C\mathrm{e}^{C_1/\tau^9}\|\varphi_0\|_{L^2(\Omega)^N}.$$

This concludes the proof of Theorem 3.4.

As in [GOT13], notice that the information of $\Delta y(\cdot,T)$ in Ω is not available in many applications, in fact, we will see that f can be recovered only from information of $\Delta y(\cdot,T)$, so formula (3.12) is useless. If we only have access to the measurements in the observatory $\omega \times (0,T)$, we can deduce the reconstruction formula of Theorem 3.5.

Our second result is the following (analogous to Theorem 1.6 in [GOT13]).

Theorem 3.5 *Let $f \in L^2(\Omega)^N$ and let $\{(\lambda_k, \varphi_k)\}_{k\geq 0}$ be the eigenvalues and $(L^2)^N$-orthonormal eigenvectors of the Stokes operator in Ω with homogeneous Dirichlet boundary conditions. Given $\sigma \in W^{1,\infty}(0,T)$, $\sigma(T) \neq 0$, such that*

$$a_k := 1 - \frac{\nu\lambda_k}{\sigma(T)} \int_0^T \mathrm{e}^{-\nu\lambda_k(T-s)}\sigma(s)\mathrm{d}s \neq 0, \tag{3.17}$$

for some $k \geq 0$, then we have the local reconstruction formula

$$P_H f_k = a_k^{-1}(\mathcal{C}_{1k} + \mathcal{C}_{2k}), \tag{3.18}$$

where P_H represents the orthogonal projector from $L^2(\Omega)^N$ onto H and $\mathcal{C}_{1k} = \mathcal{C}_1(\varphi_k)$, $\mathcal{C}_{2k} = \mathcal{C}_2(\varphi_k)$ were defined in Theorem 3.4, which only depend on the local observations of $N-1$ components of the solution of (3.1).

Proof of Theorem 3.5. To prove the Theorem 3.5 we introduce the eigenvalues and eigenvectors $(\lambda_k, \varphi_k)_{k\in\mathbb{N}}$ of the Stokes operator in Ω as follows:

$$\begin{aligned} -\Delta\varphi_k + \nabla\pi_k = {}& \lambda_k\varphi_k \quad \text{in} \quad \Omega, \\ \nabla\cdot\varphi_k = {}& 0 \qquad\ \ \text{in} \quad \Omega, \\ \varphi_k = {}& 0 \qquad\ \ \text{on} \quad \Gamma, \end{aligned} \tag{3.19}$$

and we choose φ_k orthonormal in $L^2(\Omega)^N$ such that the solution u of (3.1) admitted the representation

$$y_{\mathrm{i}}(x,t) = \sum_{k\in\mathbb{N}} \alpha_k(t)\varphi_{\mathrm{i}k}(x), \quad \forall\, \mathrm{i} = 1,\dots,N.$$

On the other hand, from (3.1) and (3.19) it is easy to check that the coefficients $\alpha_k(t)$ are given by

$$\alpha_k(t) = f_k \int_0^t \mathrm{e}^{-\nu\lambda_k(t-s)}\sigma(s)\mathrm{d}s, \tag{3.20}$$

where $f_k = (f, \varphi_k)_{L^2(\Omega)^N}$ are the unknown coefficients of the source term f, which satisfies the divergence free condition.
Additionally, by integration by parts and using (3.19) and (3.20) we obtain

$$\int_\Omega \Delta y(x,T)\cdot\varphi_k(x)\mathrm{d}x = -\lambda_k(y(\cdot,T),\varphi_k)_{L^2(\Omega)^N} = -\lambda_k\alpha_k(T). \tag{3.21}$$

Then, from (3.16), (3.20) and (3.21) we get

$$\begin{aligned}(P_H f, \varphi_k)_{L^2(\Omega)^N} := f_k = {}& -a_k^{-1}\Big(\sigma(0)\sigma(T)^{-1}\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N}(y_{\mathrm{i}}, \theta_{\mathrm{i},k}^{(T)})_{H^1(0,T;L^2(\omega))} \\ & + \sigma(T)^{-1}\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N}\int_0^T \sigma'(T-s)(y_{\mathrm{i}}, \theta_{\mathrm{i},k}^{(s)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s\Big),\end{aligned}$$

where a_k was defined in (3.17). Thus the proof of Theorem 3.5 is complete.

Remark 3.3 *In Theorem 3.5, the reconstruction formula* (3.18) *is valid if the coefficient* a_k *defined by* (3.17) *is not zero. This is true for every* $k \in \mathbb{N}$ *in the following particular cases of time dependency* σ *of the source (see [GOT13]):*

a) $\sigma := \sigma_0$ *constant.*

b) $\sigma := \sigma_1(t)$ *a non-negative and increasing function.*

c) $\sigma := \sigma_2(t) = 1 + \frac{1}{2}\cos\left(\frac{4\pi t}{T-\varepsilon}\right)$ *for* $t < T - \varepsilon$ *and* $\sigma_2 = \frac{3}{2}$ *for* $t > T - \varepsilon$.

Notice that Theorem 3.5 can be extended to the case in which a linear term $\mathrm{d}(t)y(x,t)$ is added to the main equation in (3.1), with $\mathrm{d} \in W^{1,\infty}(0,T)$, so, the new system will be given by:

$$\begin{cases} y_t - \nu\Delta y + \mathrm{d}(t)y + \nabla p = f(x)\sigma(t) & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot y = 0 & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \Gamma \times (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega, \end{cases}$$

In fact, it is known that the observability inequality (3.5) is valid in the presence of this linear term in the controlled system (3.2) and the corresponding adjoint system (3.6). Thus, using the same scheme of the proof of Theorem 3.5, it is easy to obtain for the above system the following Corollary.

Corolary 3.6 *Under the hypothesis of Theorem 3.5 and* $\mathrm{d} \in W^{1,\infty}(0,T)$*, if*

$$a_k := 1 - \frac{\nu\lambda_k}{\sigma(T)} \int_0^T \mathrm{e}^{-\nu\lambda_k(T-s)+\int_s^T \mathrm{d}(y)\mathrm{d}y} \sigma(s)\mathrm{d}s \neq 0,$$

for some $k \geq 0$*, then we have the local reconstruction formula*

$$P_H f_k = a_k^{-1}(\mathcal{C}_{1k} + \mathcal{C}_{2k} + \mathcal{C}_{3k}),$$

where P_H *represents the orthogonal projector in* $L^2(\Omega)^N$ *onto* H, $\mathcal{C}_{1k} = \mathcal{C}_1(\varphi_k)$, $\mathcal{C}_{2k} = \mathcal{C}_2(\varphi_k)$ *were defined in Theorem 3.4 and*

$$\mathcal{C}_{3k} := -\frac{\mathrm{d}(T)}{\sigma(T)} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^T \sigma(T-s)(y_{\mathrm{i}}, \theta_{\mathrm{i},k}^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.$$

3.3 Convergence of two-parametric optimal controls to null controls with one vanishing component

We also study the null controllability problem mentioned in Lemma3.1 through a sequence of optimal control problems, by introducing relaxation parameters $\alpha > 0$ and $\beta > 0$. Then, for every $\tau \in (0,T]$, let us first characterize the control of minimal norm in $L^2(0,\tau;L^2(\Omega)^N)$ by an optimal system. For $\varphi_0 \in H$ fixed, we consider the cost functional $J_{\alpha,\beta}$ defined by

$$J_{\alpha,\beta}(h) := \frac{1}{2} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^\tau \int_\omega |h_{\mathrm{i}}|^2 \mathrm{d}x\mathrm{d}t + \beta \int_0^\tau \int_\omega |h_j|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2\alpha}\|\phi(\cdot,0)\|^2_{L^2(\Omega)^N},$$

where α and β are arbitrary positive numbers, which are associated respectively to the exact final condition $\phi(\cdot,0)=0$ (with ϕ the solution of (3.2)) and the internal control with null j-th component. Next, we consider the following optimal control problem:

$$\min_{h\in L^2(0,\tau;L^2(\omega)^N)} J_{\alpha,\beta}(h). \tag{3.22}$$

In [GOT13], the authors proved a similar result of optimal control for scalar parabolic equations. The novelty here is the additional parameter β.

Theorem 3.7 *The following statements hold:*

(i) For every $\alpha>0$ and for every $\beta>0$ there exists a unique solution $h=h(\alpha,\beta)$ to (3.22) *where h is characterized by the following optimality system:*

$$\begin{cases} -\partial_t\phi-\nu\Delta\phi+\nabla\pi=h^{(\tau)}1_{\omega\times[0,\tau]} & \text{in} \quad \Omega\times(0,\tau),\\ \nabla\cdot\phi=0 & \text{in} \quad \Omega\times(0,\tau),\\ \phi=0 & \text{on} \quad \Gamma\times(0,\tau),\\ \phi(\cdot,\tau)=\varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{3.23}$$

and

$$\begin{cases} \partial_t w-\nu\Delta w+\nabla q=0 & \text{in} \quad \Omega\times(0,\tau),\\ \nabla\cdot w=0 & \text{in} \quad \Omega\times(0,\tau),\\ w=0 & \text{on} \quad \Gamma\times(0,\tau),\\ w(\cdot,0)=\dfrac{1}{\alpha}\phi(\cdot,0) & \text{in} \quad \Omega, \end{cases} \tag{3.24}$$

with

$$\begin{array}{ll} h_{\mathrm{i}}^{(\tau)}+w_{\mathrm{i}}=0 & \text{in} \quad \omega\times(0,\tau),\ \forall \mathrm{i}=1,\dots,N,\ \mathrm{i}\neq j,\\ \beta h_j^{(\tau)}+w_j=0 & \text{in} \quad \omega\times(0,\tau). \end{array} \tag{3.25}$$

(ii) When β tends to infinity and α tends to zero, we have

$$\left\{\begin{array}{l} -\quad \dfrac{\nu}{\sigma(T)}(\Delta y(\cdot,T),\varphi_0)_{L^2(\Omega)^N}-\dfrac{\sigma(0)}{\sigma(T)}\displaystyle\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N}(y_{\mathrm{i}},\theta_{\mathrm{i}}^{(T)})_{H^1(0,T;L^2(\omega))}\\ -\dfrac{1}{\sigma(T)}\displaystyle\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N}\int_0^T\sigma'(T-s)(y_{\mathrm{i}},\theta_{\mathrm{i}}^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s. \end{array}\right\}\to(f,\varphi_0)_{L^2(\Omega)^N},$$

where $\theta_{\mathrm{i}}^{(\tau)}$ is the solution of $h_{\mathrm{i}}^{(\tau)}=K^*\theta_{\mathrm{i}}^{(\tau)}$.

Proof of Theorem 3.7. The arguments are essentially based in [GOT13], [GLH08] and [Lio71], after considering the following differences:

(i) This item is checked from [Lio71]. Therefore the problem (3.22) has a unique solution $h^{(\tau)}$, which satisfies the optimality system (3.23)-(3.25).

(ii) From (3.23)-(3.25), it is easy to verify the identity:

$$\underbrace{\int_0^\tau\int_\omega\left(\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N}|h_{\mathrm{i}}^{(\tau)}|^2+\beta|h_j^{(\tau)}|^2\right)\mathrm{d}x\mathrm{d}t+\frac{1}{\alpha}\|\phi(\cdot,0)\|^2_{L^2(\Omega)^N}}_{I_2}=(w(\cdot,\tau),\varphi_0)_{L^2(\Omega)^N}. \tag{3.26}$$

Applying Young's inequality on the right-hand side of (3.26) and combining this with the observability inequality (3.5) we obtain

$$I_2 \leq \frac{a^2}{2} C_0 \mathrm{e}^{C_1/\tau^9} \sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^\tau \int_\omega |w_\mathrm{i}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2a^2} \|\varphi_0\|^2_{L^2(\Omega)^N}, \quad a > 0.$$

Choosing $a^2 = C_0^{-1}\mathrm{e}^{-C_1/\tau^9}$ and using the optimal condition $w_\mathrm{i} = -h_\mathrm{i}$, $\forall \mathrm{i} = 1, \ldots, N$, $\mathrm{i} \neq j$, we can deduce that

$$\int_0^\tau \int_\omega \left(\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} |h_\mathrm{i}^{(\tau)}|^2 + 2\beta |h_j^{(\tau)}|^2 \right) \mathrm{d}x\mathrm{d}t + \frac{2}{\alpha} \|\phi(\cdot,0)\|^2_{L^2(\Omega)^N} \leq C_0 \mathrm{e}^{C_1/\tau^9} \|\varphi_0\|^2_{L^2(\Omega)^N}, \tag{3.27}$$

where C_0, C_1 are independent of α and β. Now, since $h_\mathrm{i}^{(\tau)} 1_{\omega\times(0,\tau)}$ is uniformly bounded in $L^2(0,\tau; L^2(\Omega))$ for each $\mathrm{i} = 1, \ldots, N$, $\mathrm{i} \neq j$ and $\varphi_0 \in H$, it follows that the solution ϕ of system (3.2) is uniformly bounded in $C^0([0,\tau]; H)$ (see [Tem01], Theorem 1.1, page 172). Then, for each $n \in \mathbb{N}$ we denote by ϕ_n the solution of system (3.2) associated to $h_n^{(\tau)}$ and consider $\eta_\mathrm{i} = h_{\mathrm{i},n}^{(\tau)}$ in (3.7). Thus, we can extract subsequences $\{h_{\mathrm{i},n'}^{(\tau)}\}$, $\{\phi_{n'}\}$, and $\{\theta_{\mathrm{i},n'}^{(\tau)}\}$, with $\alpha_{n'} \to 0$ and $\beta_{n'} \to \infty$ (recall that h depends on α and β), such that

$$h_{\mathrm{i},n'}^{(\tau)} \rightharpoonup h_\mathrm{i}^{(\tau)} \quad \text{weakly in} \quad L^2(0,\tau; L^2(\omega)), \quad \theta_{\mathrm{i},n'}^{(\tau)} \rightharpoonup \theta_\mathrm{i}^{(\tau)} \quad \text{weakly in} \quad H^1(0,\tau; L^2(\Omega)),$$

and

$$\phi_{n'} \rightharpoonup \phi \quad \text{weakly in} \quad L^2(0,\tau; V), \quad \partial_t \phi_{n'} \rightharpoonup \partial_t \phi \quad \text{weakly in} \quad L^2(0,\tau; V^*),$$

where $V := \{\phi \in H_0^1(\Omega)^N : \nabla \cdot \phi = 0\}$ and V^* is the dual space of V. Therefore, using compactness argument between Banach spaces (see [Tem01], Theorem 2.1, page 184) we deduce that

$$\phi_{n'}(\cdot,0) \to \phi(\cdot,0) \quad \text{in} \quad H, \quad n' \to +\infty. \tag{3.28}$$

On the other hand, from (3.27) we have

$$\beta \|h_j^{(\tau)}\|^2_{L^2(0,\tau;L^2(\omega))} \leq C_0 \mathrm{e}^{C_1/\tau^9} \|\varphi_0\|^2_{L^2(\Omega)^N} \quad \text{and} \quad ||\phi_{n'}(\cdot,0)||_{L^2(\Omega)^N} \to 0,\ n' \to \infty,$$

this implies that $h_j^{(\tau)}$ is uniformly bounded in $L^2(0,\tau; L^2(\omega))$ and thanks to (3.28), $\phi(\cdot,0) = 0$ in Ω. Moreover, if $\beta \to +\infty$ then $h_j^{(\tau)} \to 0$ in $L^2(0,\tau; L^2(\omega))$. Finally, for fixed $\varphi_0 \in H$ we find:

$$\left\{ \begin{array}{l} -\dfrac{\nu}{\sigma(T)} (\Delta y(\cdot,T), \varphi_0)_{L^2(\Omega)^N} - \dfrac{\sigma(0)}{\sigma(T)} \displaystyle\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (y_\mathrm{i}, \theta_{\mathrm{i},n'}^{(T)})_{H^1(0,T;L^2(\omega))} \\ -\dfrac{1}{\sigma(T)} \displaystyle\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} \int_0^T \sigma'(T-s) (y_\mathrm{i}, \theta_{\mathrm{i},n'}^{(\tau)})_{H^1(0,T;L^2(\omega))} \mathrm{d}s. \end{array} \right\} \to (f, \varphi_0)_{L^2(\Omega)^N},$$

which concludes the proof of Theorem 3.7.

3.4 Numerical examples

In this section we present a two dimensional numerical implementation of the reconstruction formula (3.18) established in Theorem 3.5. In this case, the formula allows to recover the H-projection of the source for the Stokes system (3.1) with observations of one component of the solution over a subdomain $\omega \times (0,T)$. The objective is to test the feasibility of the formula for different choices of the temporal dependency of the source $\sigma(t)$ (see Remark 3.3).

Notice that we have to solve several null controllability problems (see (3.2)) and Volterra integral equations (3.11) in order to compute the projections of $f \in L(\Omega)^2$ on some given direction $\varphi_k \in H$. The numerical scheme to solve each Volterra equation is the same as in [GOT13]. On the other hand, the null-controls with one vanishing component are approximated by using the two-parameter optimal controls introduced in the previous section. More precisely, we implement the following algorithm:

Remark 3.4 *Taking into account* (3.23), *let us first introduce* $(\bar{\psi}, \bar{\pi})$ *and* $(\hat{\psi}, \hat{\pi})$, *the corresponding solutions of the following systems:*

$$\begin{cases} -\partial_t \bar{\psi} - \nu \Delta \bar{\psi} + \nabla \bar{\pi} = 0 & \text{in} \quad \Omega \times (0,\tau), \\ \nabla \cdot \bar{\psi} = 0 & \text{in} \quad \Omega \times (0,\tau), \\ \bar{\psi} = 0 & \text{on} \quad \Gamma \times (0,\tau), \\ \bar{\psi}(\cdot,\tau) = \varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{3.29}$$

and

$$\begin{cases} -\partial_t \hat{\psi} - \nu \Delta \hat{\psi} + \nabla \hat{\pi} = h^{(\tau)} 1_{\omega \times [0,\tau]} & \text{in} \quad \Omega \times (0,\tau), \\ \nabla \cdot \hat{\psi} = 0 & \text{in} \quad \Omega \times (0,\tau), \\ \hat{\psi} = 0 & \text{on} \quad \Gamma \times (0,\tau), \\ \hat{\psi}(\cdot,\tau) = 0 & \text{in} \quad \Omega. \end{cases} \tag{3.30}$$

Now, let us consider the linear operators $L : H \to L^2(0,\tau; L^2(\omega)^2)$ *and* $L^* : L^2(0,\tau; L^2(\omega)^2) \to H$ *defined by*

$$Lw(\cdot,0) := -w 1_{\omega \times [0,\tau]} \quad \text{and} \quad L^* h^{(\tau)} := -\hat{\psi}(\cdot,0),$$

where w is the solution of (3.24) with initial condition $w(\cdot,0)$ and $\hat{\psi}$ is the solution of (3.30). Furthermore, we consider the linear operator $\Lambda = L^* L : H \to H$ defined by

$$\Lambda w(\cdot,0) := -I_\beta^{(j)} \hat{\psi}(\cdot,0),$$

for either $j = 1$ or $j = 2$, where

$$I_\beta^{(1)} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_\beta^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

Thus, the solution of the optimal control system (3.23)-(3.25) is given by the unique solution of:

$$\text{Find } w(\cdot,0) \in H \text{ such that} \quad (\alpha I + I_{1/\beta}^{(j)} \Lambda) w(\cdot,0) = \bar{\psi}(\cdot,0). \tag{3.31}$$

In the previous scheme, as we have already mentioned, the null exact final condition is penalized by α and the vanishing component of the control is penalized by the second parameter β.

The finite dimensional version for the operator Λ is based on the time-space discretization of system (3.23)-(3.25). More precisely, we consider finite differences for the time discretization and a mixed finite element formulation in space using $\mathbb{P}_2$-type elements for the velocity and $\mathbb{P}_1$-type elements for the pressure which the classical finite element spaces of piecewise polynomials (see e.g. [All05], [GLH08]).

For the sake of clarity, we list all the steps involved in the reconstruction algorithm:

- Compute the matrix associated to the operator Λ: in the jth column of the matrix we put the solution of (3.23)-(3.25) with the jth basis finite element function as initial condition.
- Compute the first M eigenfunctions and eigenvectors $(\lambda_k.\varphi_k)$, $k = 1, \ldots, M$, of the Stokes system (3.19).
- For each eigenvector φ_k, compute the solution of (3.29) with initial condition $\varphi_0 = \varphi_k \in H$. Next, given the parameters α, β and $\bar{\psi}(\cdot, 0)$ solve (3.31) to obtain $w(\cdot, 0)$.
- In order to obtain the optimal control $h^{(\tau)}$, solve (3.24) with initial condition $w(\cdot, 0)$, obtained from the previous step by considering (3.25), for each $\tau \in (0, T]$.
- For each control $h^{(\tau)}$, we compute the Volterra equation $K^*\theta^{(\tau)} = h^{(\tau)}$ (recall to see the discretization of (3.11) in [GOT13]), to obtain $\theta^{(\tau)}$ for some discretized set $\tau \in (0, T]$.
- Finally, use (3.18) to find the coefficients of the source f. This complete the application of the reconstruction formula (3.18).
- Apply, if needed, an extra optimization method (3.32). See the discussion below.

In practice, we observe that the numerical results obtained with the formula (3.18) allow to detect with some accuracy the position of the source but not at all its amplitude. Therefore we implement an additional step consisting on a classical optimization algorithm that minimizes the fit between predicted and measured observations, but restricted to the frequencies associated to large amplitudes previously found. More precisely:

$$\hat{f} = \underset{g=\sum_k c_k f_k \varphi_k}{\operatorname{argmin}} \|y^m - y(g)\|^2_{H^1(0,T;L^2(\omega)^2)} + \mu \|g - f\|^2_H, \tag{3.32}$$

where y^m are the given measurements, $\mu > 0$ is some regularization parameter and f is the recovered source using the reconstruction formula (3.18) for $0 \leq k \leq M$ by adjusting the unknown coefficient f_k for which g is significant by a factor c_k.

For the numerical experiments we use the following data: we fix $\Omega = (0,1) \times (0,1)$ and $T = 1$ and $M = 38$. The observation set ω is $(0,1) \times (0.3, 0.7)$. The mesh size is $h = 1/20$ and the time step size is $\Delta t = 5 \times 10^{-3}$. The diffusion parameter is $\nu = 5 \times 10^{-2}$ and the regularization parameters are $\alpha = 5 \times 10^{-3}$ and $\beta = 15$. We consider a divergence free unknown source of the form $f = (-\partial_2 g, \partial_1 g)$, where g is a Gaussian function with amplitude $A = \frac{10}{\sqrt{2\pi}}$, center $(x_0, y_0) = (0.5, 0.8)$ and standard deviation 1×10^{-1} (see Figure 3.1 first column).

Using the functions σ_1, σ_2 mentioned in the Remark 3.6, we show in Figure 3.1 and Figure 3.2 the relative errors in $L^2(\Omega)^2$ of the Gaussian reconstructed source with respect to the projected source for both components. Here, it is important to mention that the null controls only depend on the domain Ω and the observatory ω, therefore is not necessary to recalculate them when $\sigma(t)$ is changed. In Figure 3.1, the first column shows the projection of the unknown source on H, the second column is the estimated source using formula (3.18) by observing both velocity components. The third and fourth column represent the reconstruction when a component is missing in the velocity. Finally, the last two columns represent the reconstructed source when we apply the extra optimization algorithm (3.32). The Figure 3.2 is analogous to Figure 3.1, but for another time dependency of the source $\sigma = \sigma_2(t)$ (see Remark3.3).

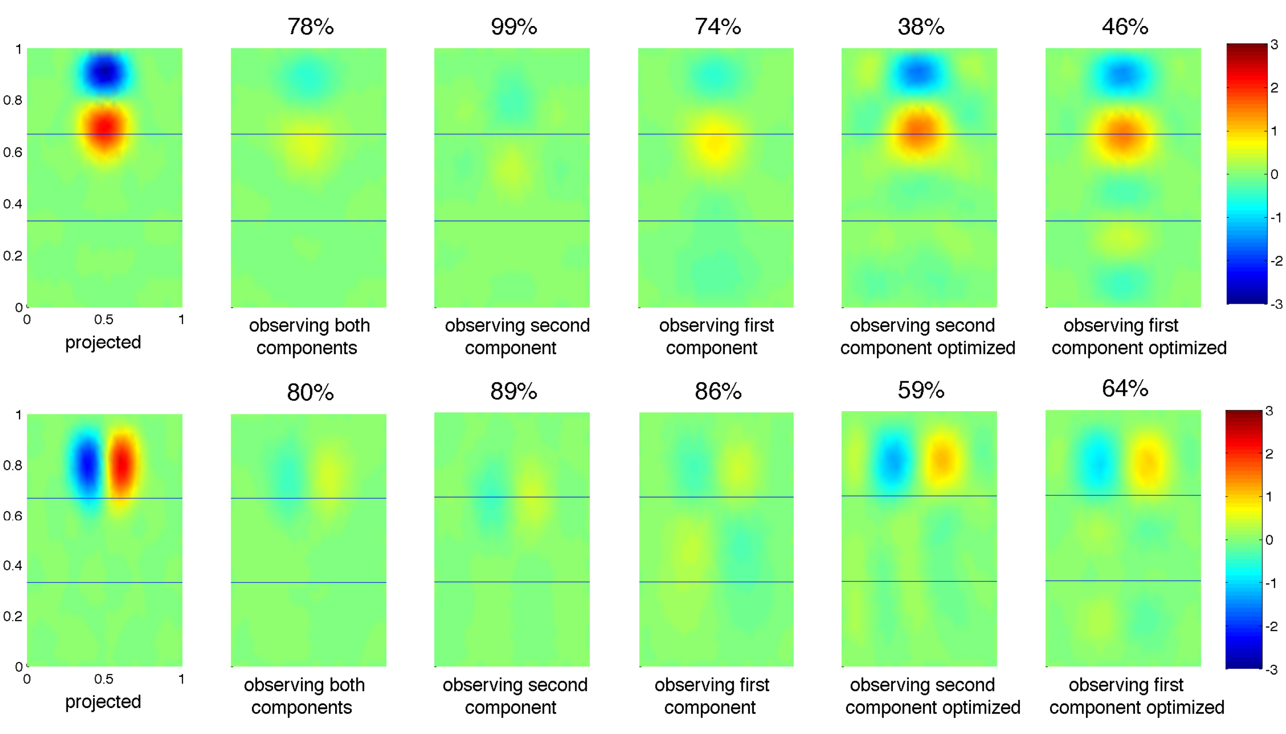

Figure 3.1: Reconstruction of both component of a divergence free source from local measurements of some components of the velocity in the observatory $\omega = (0, 1) \times (0.3, 0.7)$ using the reconstruction formula (3.18), and optimization algorithm (3.32), for the case $\sigma = \sigma_1$. The L^2 relative error of the reconstructions with respect to the projected real source is presented.

In Figure 3.3 we present the source coefficients f_k for each frequency number k in the following cases: the real one, the obtained using the reconstruction formula (3.18) and after optimization algorithm (3.32). The Figure 3.3 a) shows estimated coefficients by observing

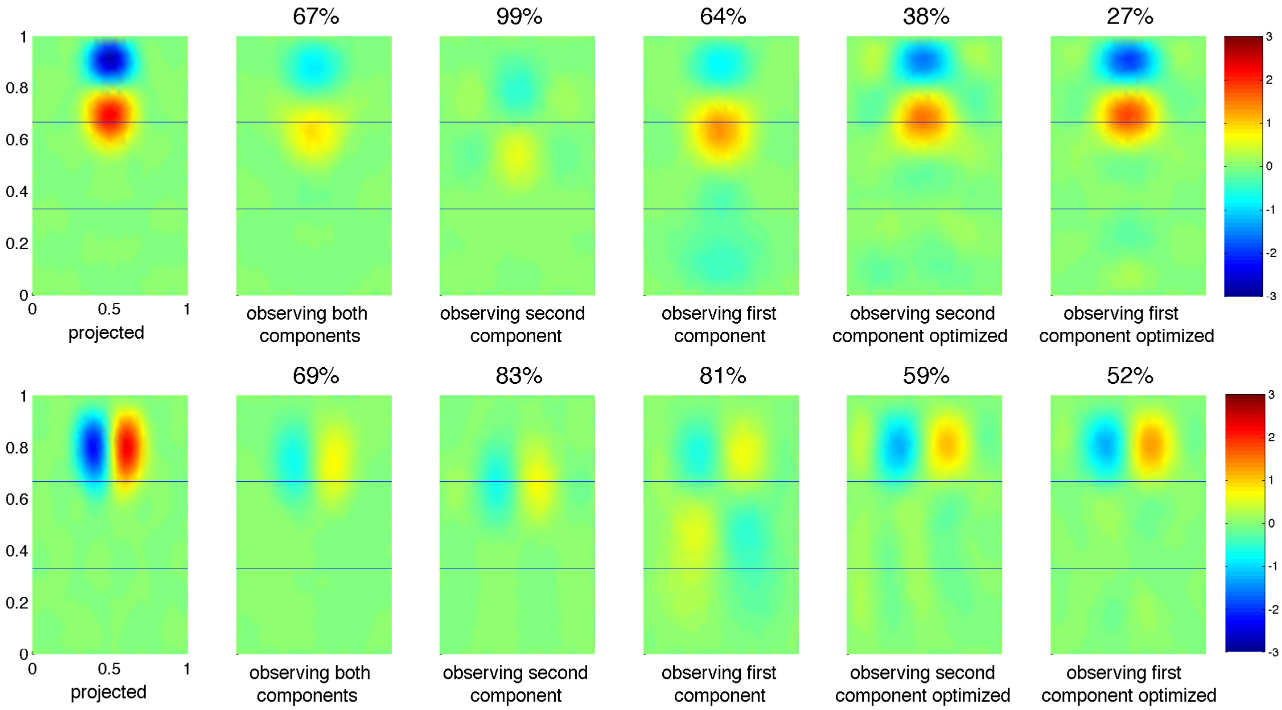

Figure 3.2: Reconstruction of both component of a divergence free source from local measurements of some components of the velocity in the observatory $\omega = (0,1) \times (0.3, 0.7)$ using the reconstruction formula (3.18), and optimization algorithm (3.32), for the case $\sigma = \sigma_2$. The L^2 relative error of the reconstructions with respect to the projected real source is presented.

the first velocity component and in Figure 3.3 b) corresponds when we observe the second component. In both cases, the optimization algorithm approximates better the coefficients, this can be clearly seen in the last two columns in Figure 3.1 and Figure 3.2.

Comments and related open problems

The strategy presented here for solving the source inverse problem could be useful for other related systems. For instance, the linear quasi-geostrophic ocean model described in [GOP11] could be also considered. However, there are not existing null-controllability results with one missing component for this type of Stokes systems. The major difficulty is the Coriolis term that is coupling the equations. The corresponding global Carleman inequalities seem difficult to prove in this case due to the weight balance that is critical in the presence of zero order terms. Thus, this inverse source problem is an open problem.

Also, it is known that local null-controllability with one missing components as presented in Lemma 3.1 is still possible to the Navier-Stokes system with Dirichlet homogeneous boundary conditions [CG13]. This motivates another open problem related to the present work, which is the extension, via linearization, of some source recovering formula in the non-linear

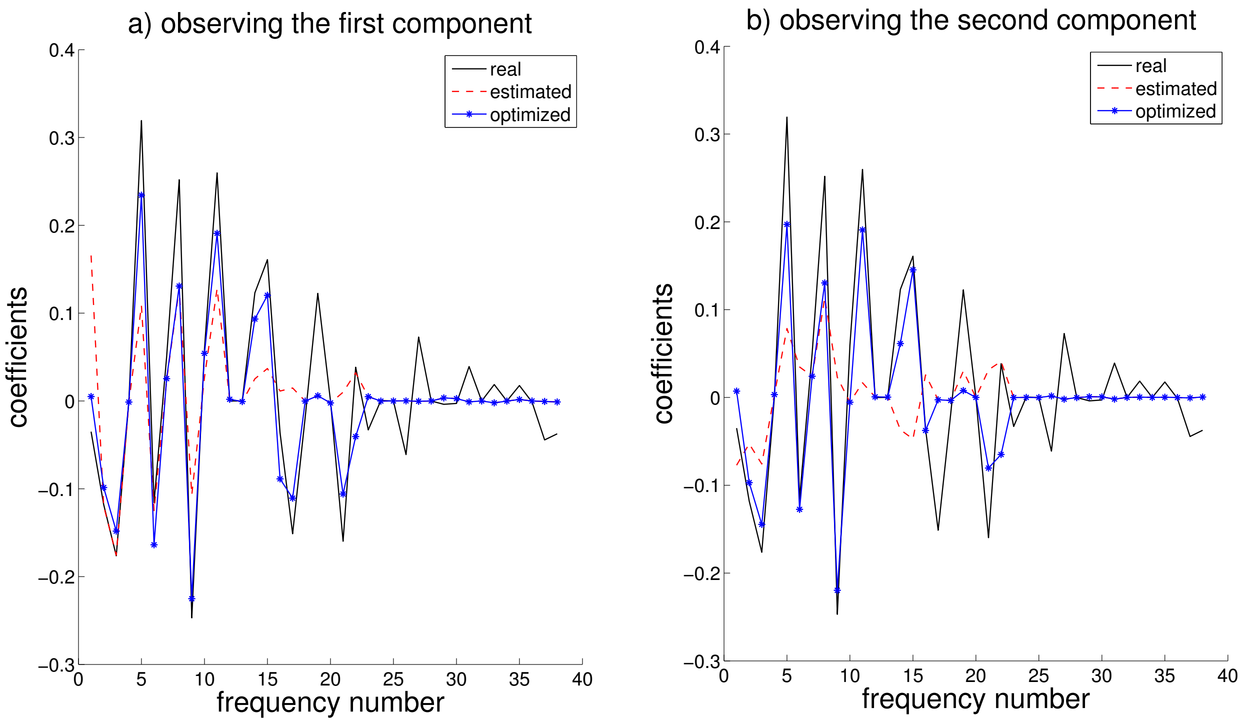

Figure 3.3: The source coefficients f_k for each frequency number k are presented in the following cases: the real one, the obtained using the reconstruction formula (3.18) and after optimization algorithm (3.32). The part a) shows the reconstruction by observing the first component of the velocity and b) shows the behavior when we observe the second component of the velocity. In this case $\sigma = \sigma_1$.

case.

Finally, when we deal with the inverse source problem for the Stokes system, we have to restrict ourselves to sources in the divergence-free space H, in order to avoid pressure measurements. The case of a source with non zero divergence is an open problem.

Chapter 4

Second inverse source problem for the Stokes system

4.1 Introduction

In this chapter we deal with an inverse problem of determining of spatially varying factor in a source term $f(x)$ of the N-dimensional Stokes system $y_t - \nu\Delta y + \nabla p = R(x)f(x)$, assuming $R(x)$ known. The main result establishes the Lipschitz stability through one component of velocity. Our result involved Carleman inequalities and degenerate elliptic operators.

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class C^∞. We will use the notation $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$ and by $n(x)$ the outward unit normal vector to Ω at the point $x \in \partial\Omega$. We consider the Stokes system for an incompressible viscous fluid flow:

$$\begin{cases} y_t - \nu\Delta y + \nabla p = F(x) & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $\nu > 0$ is a constant describing the viscosity, which by simplicity we assume that the density is one (homogeneous fluid). The density of external force that produce the movement of the fluid is

$$F(x) := R(x)f(x), \tag{4.2}$$

where $R(x) = (r_1(x), \dots, r_N(x))^t$ is a vector-valued function and $f = f(x)$ is a real-valued function.

Inverse source problem. Let $\omega \Subset \Omega \subset \mathbb{R}^N$ an arbitrary sub-domain, $0 < \theta < T$ and the velocity field y satisfying (4.1). The inverse problem is to determine $f(x)$ by observation data $y_j|_{\omega\times(0,T)}, y_j(\cdot,\theta)|_\Omega$, $y_j|_\Sigma$ for some $j \in \{1, \dots, N\}$.
In general aspects, the inverse problems of this type for the Stokes equations have not been

studied intensively. As relevant results, we refer to [CIPY13], and [POV$^+$00]. In [CIPY13], Choulli et al. proved the Lipschitz stability for linearized Navier Stokes equations with homogeneous Dirichlet boundary conditions and data in an arbitrary subset ω. The novelty of our work is the Lipschitz stability through data of one component of velocity.

We mention that the main result, Theorem 4.3, is developed in the spirit of [Kli81] and [CIPY13], and using ideas presented in [[IY98], [FCGBGP06], [Fic60], [Ole12]] and other related works. In [Kli81], the author introduce a methodology called Bukhgeim Klibanov's method the which is based on Carleman estimates to inverse problems.
The documents [Ole12] and [Fic60] treat different aspects of a general theory of second order equations with nonnegative characteristic form (also called degenerating elliptic equations or elliptic-parabolic equations), which are used in the proof of Theorem 4.3. In the context of degenerate elliptic operators, our Proposition 4.2 describe an inequality in $L^2(\Omega)$ for the Dirichlet homogeneous problem:

$$\begin{cases} L(y) \equiv a^{kj}(x)y_{x_kx_j} + b^k(x)y_{x_k} + c(x)y = h & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.3}$$

where $a^{kj}(x)\xi_j\xi_k \geq 0$ for any vector $\xi = (\xi_1, \dots, \xi_N)$. The interested reader can find more details of the problem (4.3) (existence, uniqueness, weak solutions, etc) in [Fic60],[Ole12].

4.2 Preliminary results

In this section we will present some result on Carleman inequalities and second order equations with nonnegative characteristic form, which are necessary to prove of Theorem 4.3.

4.2.1 Carleman inequalities

In order to establish the Carleman inequality, we need to define some weight functions. Let ω be a nonempty open subset of $\mathbb{R}^N$ and $\eta \in C^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega \setminus \omega}, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial\Omega. \tag{4.4}$$

The existence of such a function η is proved in [FI96b]. Then, for all $\lambda \geq 1$ we consider the following weight functions:

$$\begin{aligned} \alpha(x,t) &= \frac{\mathrm{e}^{2\lambda\eta(x)} - \mathrm{e}^{\lambda\|\eta\|_\infty}}{t(T-t)}, & \xi(x,t) &= \frac{\mathrm{e}^{\lambda\eta(x)}}{t(T-t)}, \\ \alpha_*(t) &= \min_{x\in\overline{\Omega}} \alpha(x,t), & \xi_*(t) &= \min_{x\in\overline{\Omega}} \xi(x,t), \\ \widehat{\alpha}(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t), & \widehat{\xi}(t) &= \max_{x\in\overline{\Omega}} \xi(x,t), \end{aligned} \tag{4.5}$$

In order to prove Theorem 4.3, we will use the following results, which was proved in [FCGBGP06] for a parabolic equation with Fourier boundary conditions. Let us introduce the system

$$\begin{cases} \psi_t - \Delta\psi = f_1 + \nabla \cdot f_2 & \text{in} \quad Q, \\ (\nabla\psi + f_2) \cdot n = f_3 & \text{on} \quad \Sigma, \\ \psi(\cdot, 0) = \psi_0 & \text{in} \quad \Omega, \end{cases} \tag{4.6}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. We present now this result:

Lemma 4.1 *Under the previous assumptions on f_1, f_2 and f_3, there exist $\overline{\lambda}, \sigma_1, \sigma_2$ and C, only depending on Ω and ω, such that, for any $\lambda \geq \overline{\lambda}$, any $s \geq \overline{s} = \sigma_1(\mathrm{e}^{\sigma_2\lambda}T + T^2)$ and any $\psi_0 \in L^2(\Omega)$, the weak solution to (4.6) satisfies*

$$\begin{aligned} &\iint_Q \mathrm{e}^{2s\alpha}(s\lambda^2\xi|\nabla\psi|^2 + s^3\lambda^4\xi^3|\psi|^2)\mathrm{d}x\mathrm{d}t + s^2\lambda^3 \iint_\Sigma \mathrm{e}^{2s\alpha}\xi^2|\psi|^2\mathrm{d}\sigma\mathrm{d}t \\ &\leq C\Big(\iint_Q \mathrm{e}^{2s\alpha}(|f_1|^2 + s^2\lambda^2\xi^2|f_2|^2)\mathrm{d}x\mathrm{d}t \\ &\quad + s\lambda\iint_\Sigma \mathrm{e}^{2s\alpha}\xi|f_3|^2\mathrm{d}\sigma\mathrm{d}t + s^3\lambda^4 \iint_{\omega\times(0,T)} \mathrm{e}^{2s\alpha}\xi^3|\psi|^2\mathrm{d}x\mathrm{d}t\Big). \end{aligned} \tag{4.7}$$

4.2.2 Degenerate elliptic equations

In this section we present a result about second order equations with nonnegative characteristic (also called degenerating elliptic equations). Precisely, Proposition 4.2 is the main result in this section.

The problem (4.3) was studied by Fichera in [Fic60]. In [Fic60], the author define subsets on the boundary $\partial\Omega$ and differents functions, called Fichera's functions in order to obtain a general development. We omit certain details and invite the interested reader to see [Fic60] and[Ole12].

Next, from (4.3) we introduce the notation

$$L^*(v) \equiv (a^{kj}v)_{x_kx_j} - (b^kv)_{x_k} + cv = a^{kj}v_{x_kx_j} + b^*v_{x_k} + c^*v, \tag{4.8}$$

where

$$b^* = 2a^{kj}_{x_j} - b^k, \quad c^* = a^{kj}_{x_kx_j} - b^k_{x_k} + c.$$

The following Proposition determine an estimate in the space $L^2(\Omega)$ for the problem (4.3). The arguments of the proof are based in [[Ole12], page 24, Theorem 1.2.1].

Proposition 4.2 *If $c < 0$ and $-c^* - c > 0$ in $\Omega \cup \partial\Omega$, then all function $y \in C^2(\Omega \cup \partial\Omega)$ with $y = 0$ on $\partial\Omega$ satisfies*

$$\|y\|_{L^2(\Omega)} \leq \frac{2}{\min\limits_{\Omega\cup\partial\Omega}[-c^* - c]}\|L(y)\|_{L^2(\Omega)}. \tag{4.9}$$

Remark 4.1 *We remark that the assumption $c < 0$ in Proposition 4.2 is essential and may not be replaced by the condition $c \leq 0$. This condition is based in the maximum principles [[Ole12], pag. 21].*

Proof of Proposition 4.2. The operator $L(y)$ may be written in the form

$$L(y) \equiv (a^{kj} y_{x_k})_{x_j} + (b^k - a^{kj}_{x_j}) y_{x_k} + cy.$$

Setting $b^k - a^{kj}_{x_j} =: l^k$ and using (4.8) we have

$$L^*(w) \equiv (a^{kj} w_{x_k})_{x_j} - (l^k w)_{x_k} + cw,$$

$$L(y)w - L^*(w)y = (a^{kj} w y_{x_k} - a^{kj} y w_{x_k})_{x_j} + (l^k w y)_{x_k}.$$

Integrating in Ω and applying Ostrogradsky's Theorem, we obtain

$$\int\limits_{\Omega} (L(y)w - L^*(w)y) \mathrm{d}x = - \int\limits_{\partial\Omega} [(a^{kj} w y_{x_k} - a^{kj} y w_{x_k}) n_j + (l^k w y) n_k] \mathrm{d}\sigma, \tag{4.10}$$

where n is the interior normal vector to $\partial\Omega$.
Now, for every $\delta > 0$ arbitrary, we consider the change of variable $y \to y^2 + \delta$ in (4.10) and we obtain

$$\int\limits_{\Omega} (L(y^2+\delta)w - L^*(w)(y^2+\delta)) \mathrm{d}x = - \int\limits_{\partial\Omega} [a^{kj} w (y^2+\delta)_{x_k} - a^{kj} (y^2+\delta) w_{x_k}] n_j + [l^k w (y^2+\delta)] n_k \mathrm{d}\sigma. \tag{4.11}$$

Since $y = 0$ on $\partial\Omega$ observe that $a^{kj} w (y^2 + \delta)_{x_k} n_j = 0$ on $\partial\Omega$.
On the other hands, it is easy to see that

$$L(y^2 + \delta) = 2yL(y) + c(-y^2 + \delta) + 2(y^2 + \delta) a^{kj} y_{x_k} y_{x_j}. \tag{4.12}$$

From (4.11) and (4.12) we have:

$$\begin{aligned} &\int\limits_{\Omega} [L^*(w)(y^2 + \delta) - cw(-y^2 + \delta) - 2w(y^2 + \delta) a^{kj} y_{x_k} y_{x_j}] \mathrm{d}x \\ &\qquad = 2 \int\limits_{\Omega} w y L(y) \mathrm{d}x + \delta \int\limits_{\partial\Omega} [l^k n_k w - a^{kj} w_{x_k} n_j] \mathrm{d}\sigma. \end{aligned}$$

Taking into account that $a^{kj} y_{x_k} y_{x_k} \geq 0$, $y = 0$ on $\partial\Omega$ and considering $w = -1$ it follows that

$$\int\limits_{\Omega} [L^*(-1)(y^2 + \delta) - c(-y^2 + \delta)] \mathrm{d}x \leq -2 \int\limits_{\Omega} y L(y) \mathrm{d}x - \delta \int\limits_{\partial\Omega} l^k n_k \mathrm{d}\sigma. \tag{4.13}$$

In (4.13) we now let δ approach zero. Then

$$\lim_{\delta \to 0} (y^2 + \delta)[L^*(-1) - c(y^2 + \delta)^{-1}(-y^2 + \delta)] = y^2 (L^*(-1) - c) = y^2 [-c^* - c] \geq 0 \quad \text{in } \Omega.$$

Therefore, from (4.13) we obtain that

$$\int\limits_{\Omega} y^2 \mathrm{d}x \leq \frac{2}{\min\limits_{\Omega \cup \partial\Omega} [-c^* - c]} \int\limits_{\Omega} |y||L(y)| \mathrm{d}x. \tag{4.14}$$

Applying Hölder's inequality to the integral on the right-hand side of (4.14), we obtain (4.9).

4.3 Main result

In this section we prove our the Lipschitz stability for the Stokes system (4.1), from measurements of one velocity component, and when the source F depends in space. We also discuss the difficult in the general case using the technical presented below.

Our main result is given in the following theorem.

Theorem 4.3 *Let us* $\mathrm{i}, j \in \{1, \ldots, N\}, \mathrm{i} \neq j$ *and* $0 < \theta < T$*. Let* $F(x) = R(x)f(x)$ *satisfy the conditions*

$$\sum_{\mathrm{i}=1,\mathrm{i}\neq j}^{N} (r_j(x)_{x_\mathrm{i}})_{x_\mathrm{i}} < 0, \quad r_j(x) > 0 \quad \textit{and} \quad r_\mathrm{i}(x) = 0 \quad \forall \mathrm{i} \neq j \quad x \in \Omega, \tag{4.15}$$

$$f \in C^2(\overline{\Omega}) \quad \textit{with} \quad f = 0 \quad \textit{on} \quad \partial\Omega. \tag{4.16}$$

Then there exist a constant $C = C(\overline{\Omega}, \omega, \theta, R) > 0$ *such that for all* y *satisfying* (4.1) *and* $\partial_t^k y_j \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;H^2(\Omega))$, *with* $k = 0, 1, 2$,

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\Delta^2 y_j(\cdot,\theta)\mathrm{e}^{s\alpha(\cdot,\theta)}\|_{L^2(\Omega)} + \sum_{k=0}^{2} \|(\hat{\xi})^{1/2}\mathrm{e}^{s\hat{\alpha}}\partial_t^k \Delta y_j\|_{L^2(0,T;H^{5/4}(\partial\Omega))} + \sum_{k=0}^{2} \|\xi^{3/2}\mathrm{e}^{s\alpha}\partial_t^k \Delta y_j\|_{L^2(\omega\times(0,T))} \right), \tag{4.17}$$

where $s > 0$ *is sufficiently large.*

Remark 4.2 *In Theorem 4.3, observe that* $\theta > 0$*. The case of* $\theta = 0$ *is essentially difficult, the Carleman estimates for parabolic equation must hold for* t *in a neighborhood of* θ*, namely,* $\theta - \delta < t < \theta + \delta$*, with some* $\delta > 0$*. For* $\theta = 0$*, this requires extensions of solutions for parabolic equations to* $t < 0$*, which is impossible in general. Therefore, our inverse problem with* $\theta = 0$ *is an open problem.*

Remark 4.3 *On the other hand, we observe that it suffices to consider only the case of* $\theta = T/2$*. In effect, let* $\delta = \{\theta, T-\theta\}$*, then we consider* (4.1) *in the domain* $\Omega \times (0, 2\theta)$ *instead of* $\Omega \times (0,T)$*. If* $\delta = T - \theta$*, then in* (4.1) *we make the change of the variables* $t \to t + (T - 2\theta)$ *to consider the domain* $\Omega \times (0, 2(T-\theta))$ *instead of* $\Omega \times (0,T)$*. Since* $\Omega \times (\theta - \delta, \theta + \delta) \subset \Omega \times (0,T)$*, all the conditions of Theorem 4.3 hold true.*

Proof of Theorem 4.3. Without any lack of generality, we treat the case of $j = 1$. The arguments can be easily extended to the general case.
In general, taking into account the divergence free condition of the system (4.1), we deduce

$$\Delta p = \nabla \cdot F \quad \text{in } Q. \tag{4.18}$$

The rest of the proof is divided in two steps. In step 1, we establish a Carleman inequality in which appears the observations from one component of velocity. In step 2, we connect the previous result with Proposition 4.2, which is referent to degenerate elliptic operators.

Step 1. We apply the operator Δ to the equation satisfied by y_1 and we denote $\psi := \Delta y_1$. We then have the parabolic equation

$$\psi_t - \Delta\psi = \Delta(r_1 f) - \partial_1 \nabla \cdot F \quad \text{ in } Q. \tag{4.19}$$

Using the Lemma 4.1 with $f_1 = \Delta(r_1 f) - \partial_1\nabla\cdot F$ and $f_2 = 0$, we obtain

$$\begin{aligned} s^3 \iint\limits_Q e^{2s\alpha}\xi^3|\psi|^2 dx dt \leq C\Bigg(&\iint\limits_Q e^{2s\alpha}|\Delta(r_1 f) - \partial_1\nabla\cdot F|^2 dx dt \\ &+ s\iint\limits_\Sigma e^{2s\alpha}\xi\Big|\frac{\partial\psi}{\partial n}\Big|^2 d\sigma dt + s^3 \iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^3|\psi|^2 dx dt\Bigg) \end{aligned} \tag{4.20}$$

for every $s \geq \overline{s_0}$.

By repeating this idea with $\partial_t\psi$ and $\partial_{tt}^2\psi$ in (4.19) and using (4.20), we get the following estimate:

$$\begin{aligned} s^3\sum_{k=0}^{2}\iint\limits_Q e^{2s\alpha}\xi^3|\partial_t^k\psi|^2 dx dt \leq & C\Bigg(\iint\limits_Q e^{2s\alpha}|\Delta(r_1 f) - \partial_1\nabla\cdot F|^2 dx dt \\ &+ \sum_{k=0}^{2} s\iint\limits_\Sigma e^{2s\alpha}\xi\Big|\frac{\partial^k}{\partial t^k}\frac{\partial\psi}{\partial n}\Big|^2 d\sigma dt + s^3\iint\limits_{\omega\times(0,T)} e^{2s\alpha}\xi^3|\partial_t^k\psi|^2 dx dt\Bigg) \end{aligned} \tag{4.21}$$

for every $s \geq \overline{s_0}$.

Now, taking into account that $\alpha(x,\theta) \geq \alpha(x,t)$ for $(x,t) \in Q$ and $e^{2s\alpha(x,0)} = 0$ for $x \in \overline{\Omega}$, we have

$$\begin{aligned} I_1 := &C^{-1}\int\limits_\Omega |\partial_t\Delta y_1(x,\theta)|^2 e^{2s\alpha(x,\theta)} dx \\ = &\int\limits_0^\theta \frac{d}{dt}\Big(\int\limits_\Omega \xi(x,t)^{-1}|\partial_t\Delta y_1(x,t)|^2 e^{2s\alpha(x,t)} dx\Big) dt \\ = &\int\limits_0^\theta\int\limits_\Omega (2s\xi^{-1}(\partial_t\alpha)|\partial_t\Delta y_1|^2 + (\partial_t\xi^{-1})|\partial_t\Delta y_1|^2 + 2\xi^{-1}\partial_t^2\Delta y_1\partial_t\Delta y_1)e^{2s\alpha} dx dt \\ \leq &\iint\limits_Q (2s\xi^{-1}(\partial_t\alpha)|\partial_t\Delta y_1|^2 + (\partial_t\xi^{-1})|\partial_t\Delta y_1|^2 + 2\xi^{-1}\partial_t^2\Delta y_1\partial_t\Delta y_1)e^{2s\alpha} dx dt, \end{aligned} \tag{4.22}$$

but $\partial_t\alpha(x,t)$ satisfies the estimate

$$|\partial_t\alpha(x,t)| \leq C\xi^2, \quad (x,t) \in Q,$$

so that, using (4.22) and (4.21) we deduce

$$
\begin{aligned}
& s^2 \int\limits_\Omega |\partial_t \Delta y_1(x,\theta)|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x \\
& \le C \iint\limits_Q (s^3 \xi |\partial_t \Delta y_1|^2 + s^2 |\partial_t^2 \Delta y_1|^2) \mathrm{e}^{2s\alpha} \mathrm{d}x \mathrm{d}t \\
& \le C \Bigg(\iint\limits_Q \mathrm{e}^{2s\alpha} |\Delta(r_1 f) - \partial_1 \nabla \cdot F|^2 \mathrm{d}x \mathrm{d}t \\
& \qquad + \sum_{k=0}^{2} s \iint\limits_\Sigma \mathrm{e}^{2s\alpha} \xi \Big| \frac{\partial^k}{\partial t^k} \frac{\partial \psi}{\partial n} \Big|^2 \mathrm{d}\sigma \mathrm{d}t + s^3 \iint\limits_{\omega \times (0,T)} \mathrm{e}^{2s\alpha} \xi^3 |\partial_t^k \psi|^2 \mathrm{d}x \mathrm{d}t \Bigg)
\end{aligned} \tag{4.23}
$$

for every $s \ge C$.
At this moment we have

$$
\begin{aligned}
& s^2 \int\limits_\Omega |\partial_t \Delta y_1(x,\theta)|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x \\
& \le C \sum_{k=0}^{2} \Bigg(\iint\limits_Q \mathrm{e}^{2s\alpha} |\Delta(r_1 f) - \partial_1 \nabla \cdot F|^2 \mathrm{d}x \mathrm{d}t \\
& \qquad + \| s^{1/2} (\hat{\xi})^{1/2} \mathrm{e}^{s\hat{\alpha}} \partial_t^k \Delta y_1 \|^2_{L^2(0,T;H^{1+\varepsilon}(\partial\Omega))} + s^3 \iint\limits_{\omega \times (0,T)} \mathrm{e}^{2s\alpha} \xi^3 |\partial_t^k \Delta y_1|^2 \mathrm{d}x \mathrm{d}t \Bigg),
\end{aligned} \tag{4.24}
$$

for every $\varepsilon > 0$ arbitrarily small and for every $s \ge C$.
On the other hand, applying the operator Δ to the first equation of (4.1) and using (4.18) we have

$$
\Delta(r_1(x) f(x)) - \partial_1 \nabla \cdot F(x) = \partial_t \Delta y_1(x,\theta) - \Delta(\Delta y_1(x,\theta)), \quad x \in \Omega.
$$

Let us define Lf and $\mathcal{D}_k$ for $k = 0, 1, 2$ by:

$$
L(R(x)f(x)) := \Delta(r_1(x) f(x)) - \partial_1 \nabla \cdot F(x) \quad \text{in } \Omega \tag{4.25}
$$

and

$$
\begin{aligned}
\mathcal{D}_k := & \| \Delta^2 y_1(\cdot,\theta) \mathrm{e}^{s\alpha(\cdot,\theta)} \|_{L^2(\Omega)} + \| (\hat{\xi})^{1/2} \mathrm{e}^{s\hat{\alpha}} \partial_t^k \Delta y_1 \|_{L^2(0,T;H^{5/4}(\partial\Omega))} \\
& + \| \xi^{3/2} \mathrm{e}^{s\alpha} \partial_t^k \Delta y_1 \|_{L^2(\omega \times (0,T))}.
\end{aligned} \tag{4.26}
$$

Then

$$
s^2 \int\limits_\Omega |L(R(x)f(x))|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x \le C \Big(s^2 \int\limits_\Omega |\partial_t \Delta y_1(x,\theta)|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x + s^2 \| \Delta^2 y_1(\cdot,\theta) \mathrm{e}^{s\alpha(\cdot,\theta)} \|^2_{L^2(\Omega)} \Big).
$$

Putting together (4.24) and the before inequality we deduce the following estimate:

$$
s^2 \int\limits_\Omega |L(R(x)f(x))|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x \le C \Bigg(\iint\limits_Q \mathrm{e}^{2s\alpha} |L(R(x)f(x))|^2 \mathrm{d}x \mathrm{d}t + s^3 \sum_{k=0}^{2} \mathcal{D}_k^2 \Bigg), \tag{4.27}
$$

for every $s \geq C$. Since $\alpha(x,\theta) \geq \alpha(x,t)$ for $(x,t) \in Q$, we can absorb in (4.27) the first term on the right-hand side by the left hand side, for every $s \geq C$.

Step 2. Taking into account that the operator $L(R(x)f(x)) \equiv L_R f$ depends of the dimension, we consider the following cases.

a) *Case of* $N = 3$. The operator $L(R(x,\theta)f(x)) \equiv L_R f$ defined in (4.25) can be rewritten by

$$\begin{aligned} L_R f =& r_1(\partial_{22}^2 f + \partial_{33}^2 f) - r_2 \partial_{12}^2 f - r_3 \partial_{13}^2 f \\ & - (\partial_2 r_2 + \partial_3 r_3)\partial_1 f + (2\partial_2 r_1 - \partial_1 r_2)\partial_2 f + (2\partial_3 r_1 - \partial_1 r_3)\partial_3 f \\ & + [\partial_{22}^2 r_1 + \partial_{33}^2 r_1 - \partial_{12}^2 r_2 - \partial_{13}^2 r_3] f, \end{aligned} \tag{4.28}$$

or equivalently

$$L_R f = a^{kj} f_{x_k x_j} + b^k f_{x_k} + cf, \tag{4.29}$$

where

$$A = (a^{kj})_{k,j=1}^N := \begin{pmatrix} 0 & -r_2 & -r_3 \\ 0 & r_1 & 0 \\ 0 & 0 & r_1 \end{pmatrix}, \tag{4.30}$$

$$b^1 := -\partial_2 r_2 - \partial_3 r_3, \quad b^2 := 2\partial_2 r_1 - \partial_1 r_2, \quad b^3 := 2\partial_3 r_1 - \partial_1 r_3$$

and

$$c = \partial_{22}^2 r_1 + \partial_{33}^2 r_1 - \partial_{12}^2 r_2 - \partial_{13}^2 r_3 \equiv Hess : A.$$

From (4.15) it follows that $\xi^t A \xi \geq 0$ for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$. Furthermore, it is easy to see that $c^* = 0$.
From (4.29) and (4.16), we can apply Proposition 4.2 with $c = (\partial_{22}^2 + \partial_{33}^2) r_1 < 0$. Therefore we obtain

$$\|f\|_{L^2(\Omega)} \leq \frac{2}{\min\limits_{\Omega \cup \partial\Omega} [-Hess : A]} \|L_R f\|_{L^2(\Omega)}.$$

Multiplying by $\min\limits_{x \in \overline{\Omega}} e^{2s\alpha(x,\theta)} =: C_2$ the previous inequality and putting together with (4.27), we have

$$\begin{aligned} & C_2 \|f\|_{L^2(\Omega)}^2 + s^2 \int\limits_\Omega |L(R(x)f(x))|^2 e^{2s\alpha(x,\theta)} dx \\ & \leq C \left(\iint\limits_Q e^{2s\alpha} |L(R(x)f(x))|^2 dx dt + s^3 \sum_{k=0}^2 \mathcal{D}_k^2 \right) + C_2 C(A) \|L_R f\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.31}$$

Taking $s > 0$ sufficiently large we can absorb the last term on the right-hand side onto the left-hand side. Thus the proof in the case $N = 3$ is complete.

b) *Case of* $N = 2$. The arguments presented until (4.27) are not dependent on the dimension. However, in this case the operator $L(R(x)f(x)) \equiv L_R f$ is given by

$$L_R f = r_1 \partial_{22}^2 f - r_2 \partial_{12}^2 f - (\partial_2 r_2)\partial_1 f + (2\partial_2 r_1 - \partial_1 r_2)\partial_2 f + [\partial_{22}^2 r_1 - \partial_{12}^2 r_2] f, \tag{4.32}$$

or equivalently

$$L_R f = a^{kj} f_{x_k x_j} + b^k f_{x_k} + cf,$$

where

$$\tilde{A} = (a^{kj})_{k,j=1}^N := \begin{pmatrix} 0 & -r_2 \\ 0 & r_1 \end{pmatrix},$$

and

$$b_1 := -\partial_2 r_2, \quad b_2 := 2\partial_2 r_1 - \partial_1 r_2, \quad c \equiv (Hess : \tilde{A}) = \partial_{22}^2 r_1.$$

In this case we also have $c^* = 0$. Then, using Proposition 4.2 with $c = (Hess : \tilde{A}) < 0$ we deduce

$$C_2 \|f\|_{L^2(\Omega)}^2 + s \int_\Omega |L(R(x)f(x))|^2 \mathrm{e}^{2s\alpha(x,\theta)} \mathrm{d}x$$
$$\leq C \sum_{k=0}^{2} \left(\iint_Q \mathrm{e}^{2s\alpha} |\partial_t^k L(R(x)f(x))|^2 \mathrm{d}x\mathrm{d}t + s^3 \mathcal{D}_k^2 \right) + C_2 C(\tilde{A}) \|L_R f\|_{L^2(\Omega)}^2.$$

Taking $s > 0$ sufficiently large ($s > CC_2C(\tilde{A})$) we can absorb the first and last term on the right-hand side onto the left-hand side.
This finishes the proof of Theorem 4.3.

Comment. In theorem 4.3, the hypothesis (4.15) allows us to obtain a second order operator with nonnegative characteristic form. However, in the case general of the operator $L_R f$ such that is described in (4.28) or (4.32), the condition $\xi^T A \xi \geq 0$ does not true for every $\xi \in \mathbb{R}^N$.
There exists other path in order to obtain in the same sense as above an inverse source problem for the system (4.1), where now the source is $F(x,t) = R(x,t)f(x)$ in Q, where $R(x,t)$ is a known vector field and $f(x)$ unknown. However, this way involved concepts in degenerate Sobolev spaces (see Appendix and references therein), which impose additional conditions on $R(x,t)$ and even difficult to check.
Finally, we comments that the inverse source problem for the Stokes system (4.1) with source $F(x,t) = R(x,t)f(x)$ from local and missing velocity measurements, is an open problem.

Appendix A

Degenerate Sobolev spaces

A.1 Introduction

To illustrate briefly a notion of degenerate Sobolev space, we recall that $w \in H^{1,2}(\Omega)$ is a weak solution of

$$Lw = g \quad \text{in } \Omega \tag{A.1}$$

where $L = \nabla^t Q(x) \nabla$ and $g \in L^2(\Omega)$, provided

$$-\int_\Omega \nabla^t v(x) Q(x) \nabla w(x) \mathrm{d}x = \int_\Omega v(x) g(x) \mathrm{d}x \tag{A.2}$$

for all $v \in Lip_c(\Omega)$, the space fo Lipschitz functions with compact closure in Ω. On the other hands, we would like to define a *large* Sobolev space than $H^{1,2}(\Omega)$ for which the integrals in (A.2) make sense (exploiting tha fact that $Q(x)$ may degenerate), but for which the calculus necessary for the proof the regularity continues to hold. One important feature in the classical case is that Lipschitz, or even smooth, functions are dense in $H^{1,2}(\Omega)$, and this density permits the transfer of the required calculus in $H^{1,2}(\Omega)$. There are thus two natural approaches in the literarure. One is denoted $H^{1,2}_\chi$ where χ is a collection of vector fields, and uses weak derivatives defined via integration by parts, in which a calculus os problematic, and the other is denoted $\mathcal{W}^{1,2}_Q$ where Q defined a general quadratic form, and uses strong derivatives defined by taking strong limits of Lipschitz functions, which inherits a calculus by continuity.The denegerate Sololev space $H^{1,2}_\chi$ defined using weak derivatives has at least two advantages over the degenerate Sobolev space $\mathcal{W}^{1,2}_Q$ defined using strong derivatives:

a. Membership in $H^{1,2}_\chi$ is easily decided using the definition of weak derivatives, while membership in $\mathcal{W}^{1,2}_Q$ is difficult to decide using Cauchy sequences,
b. The natural bounded mao from $H^{1,2}_\chi$ to L^2 is one-to-one while the corresponding map from $\mathcal{W}^{1,2}_Q$ to L^2 may not be-i.e. derivatives in $\mathcal{W}^{1,2}_Q$ are not uniquely determined by the L^2 component, whereas they are in $H^{1,2}_\chi$,

while the space $\mathcal{W}^{1,2}_Q$ has at least one crucial advantages over $H^{1,2}_\chi$:

c. There is a calculus available for the elements in $\mathcal{W}^{1,2}_Q$ that is inherited by continuity from the calculus for the dense subpace of Lipschitz functions, while such a calculus in generally problematic in $H^{1,2}_\chi$.

In [SW10] the authors proved that these spaces always coincide in dimension $n = 1$ whenever the are both defined, also they proved that $\mathcal{W}^{1,2}_Q$ is naturally embedded in $H^{1,2}_\chi$ (provided χ is such that $H^{1,2}_\chi$ can be defined), and as a consequence gradiants are uniquely determined in $\mathcal{W}^{1,2}_Q$. In [CRW13] showed that $\mathcal{W}^{1,2}_Q$ and $H^{1,2}_\chi$ coincide in higher dimensions for a collection of Lipschitz vector fields.

A.2 Some results in linear degenerate operators

Let $\hat{\Omega} \subset \mathbb{R}^3$ be a open and $\Omega \subset \hat{\Omega}$. In this section we mention definitions and results to second order equations with nonnegative characteristic, specifically we mention existence and spectral properties for weak solutions the second order non-elliptic linear Dirichlet problem of the form

$$\begin{aligned} Xu = \nabla' Q(x)\nabla u + \mathbf{HR}u + \mathbf{S'G}u + Fu &= \tilde{f} + \mathbf{T'g} \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{A.3}$$

where $Q = Q(x)$ denote a bounded nonnegative definite symmetric measurable matrix defined on $\hat{\Omega} \times \mathbb{R}^3$ and **H**,**G**,**R**,**S**,**T** are functions and vector fields suitable. Moreover, $\mathcal{Q}(x, \xi) = \xi' Q(x)\xi$ represent the quadratic form related to Q, this is:

i) $0 \leq \mathcal{Q}(x, \xi)$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in \hat{\Omega}$. Note that $\mathcal{Q}(x, \xi)$ may vanish for non-zero $\xi \in \mathbb{R}^3$.
ii) There is a $C_0 > 0$ so that $\mathcal{Q}(x, \xi) \leq c_0|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in \hat{\Omega}$.

Given a locally integrable to $\mathcal{Q}(x, \xi) = \xi' Q(x)\xi$ on $\hat{\Omega}$,.i.e.

$$\int_L \|Q(x)\|\mathrm{d}x < \infty \quad \text{for all compact } L \subset \hat{\Omega},$$

where $\|Q\|$ is the operator norm on 3×3 matrices (all norms on a finite dimensional space are equivalent), we can define the form-weighted vector-valued L^2 space $\mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ as consisting of all measurable $\mathbb{R}^3$-valued functions $\mathbf{v}(x) = (v_1(x), v_2(x), v_3(x))$, $x \in \hat{\Omega}$, satisfying

$$\|\mathbf{v}\|_{\mathcal{L}^2(\hat{\Omega},\mathcal{Q})} = \Big(\int_{\hat{\Omega}} \mathcal{Q}(x, \mathbf{v}(x))\mathrm{d}x\Big)^{1/2} < \infty \tag{A.4}$$

Remark A.1 *We suppose as usual that $\mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ consists of equivalent classes. In [SW10] is proved that the linear space $\mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ is complete with respect to the norm* (A.4)*, and is in fact a Hilbert space with respect to the associated iiner product*

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{\mathcal{L}^2(\hat{\Omega},\mathcal{Q})} = \int_{\hat{\Omega}} \boldsymbol{v}(x)' Q(x)\boldsymbol{w}(x)\mathrm{d}x. \tag{A.5}$$

Definition A.1 *Let $\mathcal{Q}$ be a locally integrable quadratic form on $\hat{\Omega}$. Define nonnegative functional (possibly infinite) $\|w\|_{\mathcal{Q}}$ on the linear space $Lip(\hat{\Omega})$ by*

$$\|w\|_{\mathcal{Q}} = \left(\|w\|^2_{L^2(\hat{\Omega})} + \|\nabla w\|^2_{\mathcal{L}^2(\hat{\Omega},\mathcal{Q})} \right), \quad w \in Lip(\hat{\Omega}) \tag{A.6}$$

We then define the degenerate Sobolev space $W^{1,2}_{\mathcal{Q}}$ as the completion of the linear space

$$Lip_{\mathcal{Q}}(\hat{\Omega}) = \{w \in Lip(\hat{\Omega}) : \|w\|_{\mathcal{Q}} < \infty\} \tag{A.7}$$

in the metric $\mathrm{d}(v, w) = \|v - w\|_{\mathcal{Q}}$.

Remark A.2 *In the case that $\mathcal{Q}$ and $\hat{\Omega}$ are bounded, we can equivalently define $W^{1,2}_{\mathcal{Q}}$ as the completion of C^1 in the metric $\mathrm{d}(w, v) = \|v - w\|_{\mathcal{Q}}$. Indeed, this follows inmediately from the fact that $C^1(\hat{\Omega})$ is dense in the classical Sobolev space $H^{1,2}(\Omega)$, so that given $w \in Lip(\hat{\Omega}) \subset H^{1,2}(\hat{\Omega})$ and $\varepsilon > 0$, we can find $v \in C^1(\hat{\Omega})$ with*

$$\|v - w\|_{W^{1,2}_{\mathcal{Q}}} \leq C\|v - w\|_{H^{1,2}(\hat{\Omega})} < \varepsilon.$$

By construction $W^{1,2}_{\mathcal{Q}}$ is a Banach space of equivalence classes of Cauchy sequences in $Lip_{\mathcal{Q}}(\hat{\Omega})$. If W= $\{w_k\}_{k=1}^{\infty}$ a is a Cauchy sequence of $Lip_{\mathcal{Q}}(\hat{\Omega})$ functions, i.e. $w_k \in Lip(\hat{\Omega})$ and

$$\|w_k - w_l\|_{W^{1,2}_{\mathcal{Q}}} \to 0 \quad \text{as } k, l \to \infty, \tag{A.8}$$

then there are elements (depending only on the equivalent classs) in $W^{1,2}_{\mathcal{Q}}, w \in L^2(\hat{\Omega})$ and $\mathbf{v} \in \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ such that $w_k \to w$ in $L^2(\hat{\Omega})$ and $\nabla w_k \to \mathbf{v}$ in $\mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$. The pair $(w, \mathbf{v}) \in L^2(\hat{\Omega}) \times \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ represents the equivalence class containing the Cauchy sequence W in the space $W^{1,2}_{\mathcal{Q}}$, and provides a Hilbert space isomorphism from $W^{1,2}_{\mathcal{Q}}$ to a closed subspace $W^{1,2}_{\mathcal{Q}}$ of $L^2(\hat{\Omega}) \times \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ by sending the equivalence class of W to $(w, \mathbf{v})$. it is realization $W^{1,2}_{\mathcal{Q}}$ of the degenerate Soboles space $W^{1,2}_{\mathcal{Q}}$ that we will use often in the general setting.

However, the vector-valued function $\mathbf{v} \in \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ is not in general uniquely determined by $w \in L^2(\hat{\Omega})$ if $(w, \mathbf{v}) \in L^2(\hat{\Omega}) \times \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$. In other words, if P is the Hilbert space projection of $L^2(\hat{\Omega}) \times \mathcal{L}^2(\hat{\Omega}, \mathcal{Q})$ onto $L^2(\hat{\Omega})$, then the restriction to $W^{1,2}_{\mathcal{Q}}$ is not in general one-to-one (see [FKS82] for a well known example).

The space $W^{1,2}_{Q,0}$ is obtained in a similar manner, but in this case we complete the set $Lip_0(\Omega)$, the set Lipschitz functions having compact support in Ω with respect to the norm (A.6).

For clarity, we always write $QH^1(\Omega)$ and $QH^1_0(\Omega)$ in place of $W^{1,2}_{Q}(\Omega)$ and $W^{1,2}_{Q,0}(\Omega)$ respectively, taking isomorphism in context. We adopted this notation in lieu of $W^{1,2}_{Q}(\Omega)$ and $W^{1,2}_{Q,0}(\Omega)$, as is used in [SW10], [CRW13], in order to agree with classical literature. See for example [MR15], where it is convention that $''W''$ spaces refer to Sobolev spaces defined with

respect to distributional derivatives. Moreover, in all of our developments we will denote the vector valued function $\overrightarrow{g}$ of the pair $(w, \overrightarrow{g}) \in QH^1(\Omega)$ by writing $\overrightarrow{g} = \nabla w$, and we will refer to it as the gradiant part (or simple the gradiant) of w and we will often abused notation by writing $w \in QH^1(\Omega)$ in place of $(w, \nabla w) \in QH^1(\Omega)$.
We also mention that is possible to introduce definitions and make similar considerations for the spaces $QH^{1,p}(\Omega)$, $QH_0^{1,p}(\Omega)$ for $1 \leq p < \infty$, even in the case $|Q(x)|$ is locally nobounded. For a complete discussion see [[SW10], [CRW13], [MR15]].

Notation. Consider a vector field

$$W(x) = \sum_{i=1}^{n} w_i(x) \frac{\partial}{\partial x_i} = (w_1(x, \ldots, w_n(x))) \cdot \nabla.$$

If u is a real valued function on $\mathbb{R}^n$ and ν is a vector in $\mathbb{R}^n$ we adopt the notation

$$Wu = \sum_{i=1}^{n} w_i \frac{\partial u}{\partial x_i}, \quad \langle \nu, W \rangle = \sum_{i=1}^{n} w_i \nu_i,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. The formal adjoint $W'(x)$ of the field $W(x)$ is denoted by

$$W'(x)u := -\text{div}(w_1(x)u(x), \ldots, w_n(x)u(x)) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(w_i(x)u(x)).$$

A vector field $W(x)$ as above is always identified with the vector valued function $(w_1(x), \ldots, w_n(x))$ and is said to be **subunit** respect to the matrix Q in Ω if

$$\Big(\sum_{i=1}^{n} w_i(x)\xi_i\Big)^2 \leq \langle \xi, Q(x)\xi \rangle \tag{A.9}$$

for every $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega$.

Remark A.3 *If a vector field $W(x)$ si subunit with respect to the matrix $Q = Q(x)$ in Ω we will simply refer to it as a "subunit vector field " with the set Ω and matrix Q taken in context.*
Given $N \in \mathbb{N}$ an N-tuple $\boldsymbol{W} = (W_1, \ldots, W_N)$ of vector fields and an $\mathbb{R}^N$-valued function $\boldsymbol{G} = (g_1, \ldots, g_N)$, $\boldsymbol{WG}$ denotes the inner product "of $\boldsymbol{W}$ and $\boldsymbol{G}$",i.e.

$$\boldsymbol{WG} = \sum_{i=1}^{N} W_i(x) g_i(x).$$

Lastly, if u is a real valued function,

$$\boldsymbol{GW}u = \sum_{i=1}^{N} g_i(x) W_i(x) u(x), \quad \boldsymbol{W'}(\boldsymbol{G}u) = \sum_{i=1}^{N} W_i'(x)(g_i(x)u(x)). \tag{A.10}$$

As in the elliptic case presented in [GT15], a negativity condition for the lower order terms **G,S** and F of X will be required.

Definition A.2 *Let us* $\Omega \subset \hat{\Omega}$ *open bounded domain in* $\mathbb{R}^n$*. We say that* X *satisfies a negativity condition if and only if*

$$\int_\Omega (Fw + \boldsymbol{GS}w)\mathrm{d}x \leq 0 \tag{A.11}$$

for all $w \in Lip_0(\Omega)$ *satisfying* $w(x) \geq 0$ *in* Ω*.*

Remark A.4 *The condition* (A.11) *is the key property that allows the application of the Fredholm Alternative enabling one to conclude existence of weak solution to the problem* (A.3)*, see [Rod11].*
This can also be seen in the elliptic case. For example, setting $\boldsymbol{G} = \boldsymbol{H} = \overrightarrow{0}$*,* $\boldsymbol{g} = 0$*,* $F = c$ *for a fixed constant c and* $Q(x) = I\mathrm{d}$*, equation* (A.3) *becomes the elliptic equation*

$$\Delta u + cu = \tilde{f}.$$

Here, the negativity condition (A.11) *becomes* $c \leq 0$ *which is sufficient for the existence of weak solutions to equations of this type, see [GT15]. Lastly, the condition* (A.11) *can differ of the presented in [GT15] by a negative sign, but they are equivalent. This is due to the usage of the formal adjoint* ***S'*** *of the vector field* ***S*** *in* (A.3)*. This term appears as* $-S'$ *in [GT15].*

Definition A.3 *Let* $\Omega \subset \hat{\Omega}$*. A second order operator* X *of the form*

$$X = \nabla' Q(x)\nabla + \boldsymbol{HR} + \boldsymbol{S'G} + F \tag{A.12}$$

is said to be of the subelliptic class related to $(\hat{\Omega}, Q, \Omega)$ *if and only if*

- *i)* $Q(x)$ *is a bounded measurable non-negative definite symmetric matrix defined in* Ω *satisfying* (A.11)*.*
- *ii)* ***R,S*** *are, for some* $N \in \mathbb{N}$*,* N*-tuples of first order vector fields subunit with respect to* Q *in* Ω*,*
- *iii)* ***H,G*** *are measurable* $\mathbb{R}^N$*-valued functions defined in* Ω*,* F *is a real valued measurable function defined in* Ω *and*
- *iv)* ***S,G***$, F$ *satisfy the negativity condition* (A.11)*.*

We list the Poincaré and Sobolev inequalities adapted to the matrix Q in order to describe Theorem A.4.

The local Poincaré Inequality. We say that the local Poncaré inequality of order p holds if there are constants $C > 0$ and $\mathbf{b} \geq 1$ so that for every ρ-ball $B(y, r)$ centered in $\hat{\Omega}$ with $\mathbf{b}r \in (0, r_1(y))$ the inequality

$$\Big(\frac{1}{|B_r|}\int_{B_r} |f - f_{B_r}|^p \mathrm{d}x\Big)^{1/p} \leq Cr\Big(\frac{1}{|B_{\mathbf{b}r}|}\int_{B_{\mathbf{b}r}} |\sqrt{Q}\nabla f|^p \mathrm{d}x\Big)^{1/p} \tag{A.13}$$

holds for all $f \in Lip_{loc}(\hat{\Omega})$. Notice that a continuity argument allows one to extend (A.13) to hold for all pairs $(f, \nabla f) \in QH^{1,p}(\hat{\Omega})$.

The Global Sobolev Inequality. For an open set $\Omega \subset \hat{\Omega}$ with $\overline{\Omega} \subset \hat{\Omega}$, we say that the global Sobolev inequality holds on Ω holds if there are postitive constants $C > 0$ and $\sigma > 1$ such that

$$\Big(\int_\Omega |f|^{2\sigma}\mathrm{d}x\Big)^{\frac{1}{2\sigma}} \le C\Big(\int_\Omega |\sqrt{Q}\nabla f|^2\mathrm{d}x\Big)^{\frac{1}{2}} \tag{A.14}$$

holds for all $f \in Lip_0(\Omega)$.

The Global Poincaré with Gain ω. For an opens subset Ω of $\hat{\Omega}$ satisfying $\overline{\Omega} \subset \hat{\Omega}$ we say that the global Poncaré inequality with gain $\omega > 1$ holds on Ω if there are contants $C > 0$ and $\omega > 1$ such that

$$\Big(\int_\Omega |f - f_\Omega|^{2\omega}\mathrm{d}x\Big)^{\frac{1}{2\omega}} \le C\Big(\int_\Omega |\sqrt{Q}\nabla f|^2\mathrm{d}x\Big)^{\frac{1}{2}} \tag{A.15}$$

holds for all $f \in Lip_Q(\Omega)$.

Remark A.5 1. *If the global Poincaré inequality* (A.15) *holds, then Holder's inequality implies that the Global Weak Poincaré Inequality gain $\omega > 1$:*

$$\Big(\int_\Omega |f|^{2\omega}\mathrm{d}x\Big)^{\frac{1}{2\omega}} \le C\Big(\int_\Omega |\sqrt{Q}\nabla f|^2\mathrm{d}x + \int_\Omega |f|^2\mathrm{d}x\Big)^{\frac{1}{2}} \tag{A.16}$$

also holds for all $f \in Lip_Q(\Omega)$.

2. *In the elliptic case ($Q(x) = I\mathrm{d}$), inequality of the form* (A.15) *and* (A.16) *are proved when the boundary of Ω is sufficiently regular. For example, $\partial\Omega \in C^{0,1}$ is used in [GT15] for such purposes. See [[GT15], Theorem 7.26] and related discussions.*
3. *In the elliptic case, where $Q(x) = I\mathrm{d}$, the classical Sobolev inequality has the form* (A.14), *for $n \ge 3$, where $\sigma = \frac{n}{n-2}$ and $C = \dfrac{2(n-1)}{\sqrt{n}(n-2)}$, see [GT15].*

The above inequalities are assume on quasimetric balls given by a quasimetric $\rho(x, y)$ defined in $\hat{\Omega}$ and upper semicontinuous in the second variable. The quasimetric ball of radius $r > 0$ centred at $x \in \hat{\Omega}$ is given by

$$B_r(x) = \{y \in \hat{\Omega} : \rho(x, y) < r\}.$$

The principal result of this section assume that the pair $(\hat{\Omega}, \rho)$ is a homogeneous space. As in [SW06], a pair $(\hat{\Omega}, \rho)$ is a homogeneous space if ρ is a above and the collection of quasimetric balls $\{B_r(y)\}_{r>0; y\in\hat{\Omega}}$ satisfies a doubling condition with respect to Lebesgue measure. That is, there are constants $c_2 > 1, C_2 > 0$ so that

$$|B_{c_2 r}(y)| \le C_2|B_r(y)|$$

for all $y \in \hat{\Omega}$ and $r > 0$.

Theorem A.4 *Let $(\hat{\Omega}, \rho)$ be a geometric homogeneous space and let Ω be a bounded domain such that $\overline{\Omega} \subset \hat{\Omega}$. Assume that the Poincaré inequality* (A.13) *holds with $p = 2$ and that the global Sobolev inequality* (A.14) *with gain $\sigma > 1$ holds. Let X be a second order linear degenerate subelliptic operator with rough coefficients as in* (A.3). *Assume that $F \in L^t(\Omega)$ with $t > \sigma'$ and $\boldsymbol{G}, \boldsymbol{H} \in [L^q(\Omega)]^N$ with $q > 2\sigma'$. Then each of the following hold.*

1) *There exists an at most countable set* $\Sigma \subset \mathbb{R}$ *such that the* X*- Dirichlet problem*

$$\begin{cases} Xu = \lambda u + \tilde{f} + \boldsymbol{T}\text{'}\boldsymbol{g} & in \quad \Omega \\ u = 0 & in \quad \partial\Omega \end{cases} \tag{A.17}$$

admits a unique weak solution $u \in H_{QH_0^1(\Omega)}$ *for every* $\tilde{f} \in L^2(\Omega)$*, every* $K \in \mathbb{N}$*, every* K*-tuple* $\boldsymbol{T}$ *of subunit fields and every* $\boldsymbol{g} \in [L^2(\Omega)]^K$ *if and only if* $\lambda \notin \Sigma$.

2) *If* Σ *is infinite, its elements can be arranged in a monotone sequence diverging to* $+\infty$.

3) *If* $\lambda \notin \Sigma$ *there exists a constant* $C = C(\lambda, \Omega, \boldsymbol{G}, \boldsymbol{H}, F) > 0$ *such that*

$$\|u\|_{QH_0^1(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \sqrt{K}\|\boldsymbol{g}\|_{L^2(\Omega)}\Big) \tag{A.18}$$

whenever $f \in L^2(\Omega)$*,* $K \in \mathbb{N}$*,* $\boldsymbol{T}$ *is a* K*- tuple of subunit vector fields,* $\boldsymbol{g} \in [L^2(\Omega)]^N$ *and* $u \in QH_0^1(\Omega)$ *is a weak solution of* (A.3).

4) *If* $\lambda \in \Sigma$*, let* $N \subset QH_0^1(\Omega)$ *be the subspace of weak solution of the* X*-Dirichlet problem*

$$\begin{cases} Xu = \lambda u & in \quad \Omega \\ u = 0 & in \quad \partial\Omega \end{cases}$$

and $N^* \subset QH_0^1(\Omega)$ *be a subspace of weak solutions of the adjoint problem*

$$\begin{cases} X^*u = \lambda u & in \quad \Omega \\ u = 0 & in \quad \partial\Omega \end{cases}$$

Then $1 \leq \dim N = \dim N^* < \infty$ *and problem* (A.3) *admits a weak solution* $u \in QH_0^1(\Omega)$ *if and only if*

$$\int_\Omega \tilde{f}v + \boldsymbol{g}\boldsymbol{T}v \,\mathrm{d}x = 0 \quad \text{for all } v \in N^*.$$

5) *If* X *satisfies negativity condition (neq), see Definition bla, then* $\Sigma \subset (0, \infty)$.

1. *If* X *is self-adjoint (that is, if* $\boldsymbol{HR} = \boldsymbol{GS}$ *almost everywhere in* Ω*), then all eigenvalues of* X *are real,* Σ *is infinite and we have the following variational characterization of the eigenvalues of* X*:*

$$\lambda_1 = \min \Sigma = \min_{u \in QH_0^1(\Omega) - \{(0,\boldsymbol{h})\}} \frac{\mathcal{L}(u,u)}{\int_\Omega u^2 \mathrm{d}x},$$

and there exists an eigenfunction $(u_1, \nabla u_1) \in QH_0^1(\Omega)$ *of the* X*-Dirichlet problem* (A.3) *related to the eigenvalue* λ_1 *for whom* $u_1 \geq 0$ *a.e. in* Ω*. Furthemore,*

$$\lambda_2 = \min\{\frac{\mathcal{L}(u,u)}{\int_\Omega u^2 \mathrm{d}x} : u \in QH_0^1(\Omega) - \{(0,\boldsymbol{h})\}, \int_\Omega uu_1 \mathrm{d}x = 0\},$$

with corresponding eigenfunction $(u_2, \nabla u_2) \in QH_0^1(\Omega)$ *where* u_2 *is orthogonal to* u_1 *in* $L^2(\Omega)$*. Recursively, for every* $k \in \mathbb{N}$ *and for every* $j = 1, \ldots, k-1$*,*

$$\lambda_k = \min\Big\{\frac{\mathcal{L}(u,u)}{\int_\Omega u^2 \mathrm{d}x} : u \in QH_0^1(\Omega) - \{(0,\boldsymbol{h})\}, \int_\Omega uu_j \mathrm{d}x = 0\Big\},$$

with corresponding eigenfunction $(u_k, \nabla u_k) \in QH_0^1(\Omega)$ *where* u_k *is orthogonal to* u_j *in* $L^2(\Omega)$ *for every* $j = 1, \ldots, k-1$. *Moreover,* $\lambda \in \mathbb{R}$ *is an eigenvalue if and only if* $\lambda_k = \lambda$ *for some* $k \in \mathbb{N}$. *The sequence* $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ *forms a complete orthogonal system of* $L^2(\Omega)$. *The sequence* $\{(u_k, \nabla u_k)\}_{k\in\mathbb{N}} \subset QH_0^1(\Omega)$ *is an independent system of element of* $QH_0^1(\Omega)$, *which is also a system of generators of* $QH_0^1(\Omega)$ *if and only if the projection* $\mathrm{i} : QH_0^1(\Omega) \to L^2(\Omega)$ *is injective. Finally, problem* (A.17) *is variational with associated functions defined on* $QH_0^1(\Omega)$ *by*

$$I(u) = \frac{1}{2}\mathcal{L}(u,u) - \frac{\lambda}{2}\int_\Omega u^2 \mathrm{d}x - \int_\Omega \tilde{f}u + \boldsymbol{g}\boldsymbol{T}u\mathrm{d}x.$$

Remark A.6 *Theorem A.4 is a direct consequence of spectral results for the* X*- Dirichlet problem with* X *a second order linear degenerate elliptic operator with rough coefficients described in [MR15].*

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