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Observer Design For Multidimensional Parabolic Systems ⋆

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Abstract: In this paper we present some observer designs for infinite dimensional parabolic systems, defined in multidimensional space domains. We consider two observation scenarios. (i) When the state is measured in an open (possibly small) subdomain of the space domain we obtain an exponentially convergent observer with gain for a semilinear parabolic system in arbitrary domains. (ii) When the state is measured along a line orthogonal to the sides of a rectangle in $\mathbb{R}^2$ we derive an observer for a linear parabolic system. The mathematical setting relies upon the semigroup theory for n–dimensional autonomous systems as well as on Fourier analysis.

Keywords: Observers, n–dimensional parabolic equations, Fourier analysis.

1. INTRODUCTION

The problem of designing observers for systems described by partial differential equations (PDEs) can be treated, roughly speaking, by using two different approaches, namely, early–and late–lumping designs. In the first case the PDEs are first approximated by finite dimensional systems for which the observers are designed using well-known methods for designing observers for finite dimensional systems (Deans and Lapidus, 1960; Christofides, 2012; Antoniades and Christofides, 2001; Park and Cho, 1996; Alonso et al., 2004; Marko et al., 2018). In the second approach the observers are directly designed using the PDE description of the plant, and the finite dimensional schemes are only used for numerical purposes (Smyshlyaev and Krstic, 2005; Meurer, 2013a; Boubaker et al., 1998; Ligarius and Couchouron, 1997; Curtain, 1982; Dochain, 2000; Schaum et al., 2015, 2016, 2017; Marko et al., 2018). Although the first approach is apparently more simple from the methodological point of view, since the design is reduced to known techniques for finite dimensional systems, the late-lumping approach is more natural from the system's theoretic point of view and can lead to simpler observer dynamics and more direct convergence conditions, since the approximation and the convergence questions are separated. In this paper we will consider the design of observers for semilinear parabolic systems with multiple space dimensions using a late-lumping approach.

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Note that in any of the approaches the observer design requires taking into account the kind of measurements at hand and are obviously linked with the observation equation available for the process.

Late–lumping approaches have been already employed by many authors. Among others, (Smyshlyaev and Krstic, 2005) design boundary observers using the backstepping technique, (Dochain, 2000) provide the so called asymptotic observers. In (Meurer, 2013a) extended Luenberger observers are developed, while variable structure estimation schemes are considered in (Boubaker et al., 1998). An observer design based on matrix inequalities is proposed in (Schaum et al., 2015). Observers based on nonlinear evolution equations and absolute stability are presented in (Ligarius and Couchouron, 1997; Curtain et al., 2003), and (Schaum et al., 2015, 2016, 2017) develop some kind of reduced–order observers.

Motivated by the 1–dimensional semilinear parabolic reaction systems observer design within a pointwise innovation scheme whose measurement is inside of the domain (in–domain) (Schaum et al., 2015, 2016, 2017), we deal in this paper with the problem of designing observers for a class of N–dimensional semilinear parabolic systems with internal local measurements. We consider here two different measurement scenarios: (1) For arbitrary N we consider that the state is measured on an (possibly small) open subset of the space domain. For $N = 1$ this means e.g. that we have measurements on a (small) interval of the spatial domain. In this scenario, we will consider Ω a domain of class C^2. (2) For the two dimensional case we consider the case of measuring on a line for a rectangular domain, that means we take a Lipschitz domain. For $N = 1$ this reduces to the case of having measurements on a point in the interior of the space domain. For the semilinear

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parabolic systems observer design in higher dimensions, the stability analysis relies on the study of abstract Cauchy problem for autonomous parabolic systems and spectral theory using Fourier series. As far as we know, the only results concerning to observers design for N–dimensional parabolic systems are given in Jadachowski et al. (2014); Meurer (2013b) with observations on the boundary, so our work has as goal to fill, at least partially, this gap.

The rest of the paper is organized in the following manner. In Section 2 we present the class of semilinear parabolic systems considered in this work, and the observation problems we deal with. In Section 3 we present the observer design for the N-dimensional case, when we have measurements on an open subset of the space domain. In Section 4 we present a solution to the 2-dimensional case with a rectangular domain, when the measurements consist of a line orthogonal to one of the boundaries of the rectangle. We conclude the paper with some final remarks in 6. We include some simulations at the end of the paper.

2. PROBLEM FORMULATION

In this paper, we consider Ω an open bounded subset of $\mathbb{R}^N$, $N \geq 2$ whose $\partial\Omega$ is regular enough, and we take the nonlinear heat system with Dirichlet boundary conditions. We put $Q = \Omega \times (0,\infty)$ and $\Sigma = \partial\Omega \times (0,\infty)$. Given $f : \mathbb{R} \to \mathbb{R}$ a given function regular enough, we consider

$$\begin{cases} z_t - \Delta z + f(z) = 0 & \text{in } Q, \\ z = u & \text{on } \Sigma, \\ z(\cdot,0) = z_0(\cdot) & \text{in } \Omega, \\ \mathsf{y} = m(z) & \text{in } Q, \end{cases} \tag{1}$$

where $z = z(x,t)$ represents the temperature, u is the control function that is assumed regular enough, which acts on the system through the boundary, z_0 is the initial state, f is a nonlinear function of the state z which belongs to an appropriate Sobolev space, and y is the measurement of the temperature z. Here we consider two different scenarios:

- For $N > 2$ the temperature is measured in a subdomain $\mathcal{O}$, where $\mathcal{O} \subset \Omega$ is a nonempty open subset and Ω is a domain of class C^2 (that means that the boundary $\partial\Omega$ is a $N-1$ manifold of class C^2) .
- For $N = 2$ the temperature is measured along a simple curve γ, such that $\gamma \subset \overline{\Omega}$. In particular, we consider that the domain Ω is a rectangle and that the curve γ is a segment orthogonal to one of the sides of the rectangle. In this situation the boundary $\partial\Omega$ is a Lipschitz manifold. We study here just the linear case.

3. LOCAL IN–DOMAIN OBSERVATIONS

In this section we will propose a simple observer design for a multidimensional semilinear parabolic systems, with arbitrary N, when the measurements are performed in an open domain $\mathcal{O} \subset \Omega$. That means that $m(z) = z1_{\mathcal{O}}$ in (1). In this section we assume that Ω is a domain of class C^2 and the control u is a given function regular enough (i.e. $u \in L^2(0,T; H^{1/2}(\partial\Omega))$. This design is performed by using a semigroup approach.

In this context we propose a simple observer structure, which yields a non–redundant estimation, given by:

$$\begin{cases} w_t - \Delta w + f(w) = k_0(w1_{\mathcal{O}} - m(z)) & \text{in } Q, \\ w = u & \text{on } \Sigma, \\ w(\cdot,0) = w_0(\cdot) & \text{in } \Omega. \end{cases} \tag{2}$$

We define the observation error $e := z - w$. From (1) and (2) we have that e satisfies the system

$$\begin{cases} e_t - \Delta e + f(z) - f(w) = -k_0 e 1_{\mathcal{O}} & \text{in } Q, \\ e = 0 & \text{on } \Sigma, \\ e(\cdot,0) = e_0(\cdot) & \text{in } \Omega, \end{cases} \tag{3}$$

or in an abstract setting

$$\begin{cases} e_t(t) = Ae(t) + F(z(t), w(t)), \quad \forall t \in [0,T], \\ e(\cdot,0) = e_0(\cdot), \end{cases} \tag{4}$$

where $e_0(\cdot) = z_0(\cdot) - w_0(\cdot)$ in Ω, $F(z(t),w(t)) := f(z) - f(w)$ and A is the unbounded linear operator defined by

$$A := \Delta - k_0 I 1_{\mathcal{O}}. \tag{5}$$

Our aim is to show that the estimation error $e(t)$ converges asymptotically to zero, so that the estimated state $w(t)$ converges asymptotically to the true value of the state $z(t)$.

In order to prove our result regarding the observer (2) we need to recall some functional analysis results used on this section.

3.1 Laplace operator with a potential

In this subsection we consider the elliptic operator of the form

$$-Lu = -\Delta u + V(x)u,$$

where the potential term V is an appropriate space. Here, we also consider the associated Dirichlet problem, i.e.,

$$\begin{cases} -\Delta u + V(x)u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

as well as the spectral problem to L: find λ and $\varphi \neq 0$ satisfying

$$\begin{cases} -\Delta\varphi + V(x)\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The next Proposition establishes estimates in the setting of Sobolev spaces for solutions of (6). The interested reader can find a proof in (Agmon et al., 1959), (Brezis, 2010).

Proposition 1. Let $p \in (1,+\infty)$ and $h \in L^p(\Omega)$. Assume that $V \in L^\infty(\Omega)$ and $V \geq 0$ a.e in Ω. Then, there exists a unique solution $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ of (6). Furthermore, there exists a positive constant C such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|h\|_{L^p(\Omega)}. \tag{8}$$

Now, we continue with the characterization of the principal eigenvalue of the operator $-L$. Briefly speaking, the eigenvalues of the operator $-L$ can be viewed as minimizers of a certain functional. More precisely, we have the following proposition.

Proposition 2. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$ such that $\partial\Omega \in C^2$ and $\mathcal{O}$ an open subset of Ω. Then, $-L = -\Delta + k_0 I 1_{\mathcal{O}}$ has a smallest eigenvalue which is denoted by λ_1 and characterized by the following statements:

i) The eigenvalue problem (7) has positive eigenvalues.

ii)

$$\lambda_1 = \inf_{\substack{\varphi \in H_0^1(\Omega) \\ \|\varphi\|_{L^2(\Omega)}=1}} \left(\int_\Omega |\nabla \varphi|^2 dx + k_0 \int_{\mathcal{O}} |\varphi|^2 dx \right). \quad (9)$$

iii) If $\varphi > 0$ is an eigenfunction associated to the eigenvalue λ, then $\lambda = \lambda_1$.

iv) λ_1 is the supremum of all $\lambda \in \mathbb{R}$ such that there exists $u \in H^2(\Omega)$ such that $u > 0$ in Ω and

$$-\Delta u + k_0 u 1_{\mathcal{O}} \geq \lambda u \quad \text{in } \Omega. \quad (10)$$

The proof of Proposition 2 can be done following the steps of the Lemma 2.4 in Dávila and Dupaigne (2003). Also a similar result but for Navier-Stokes is proved in Barbu and Lefter (2003).

Remark 3.1. For the stationary system associated to (3), if we assume

$$h \in L^p(\Omega) \cap Lip_0(\Omega), \; p \in (1, \infty),$$

$$V(x) := k_0 1_{\mathcal{O}}(x) \in L^\infty(\Omega) \quad \text{and} \quad k_0 \geq 0,$$

then, there exists a unique solution of $-Ae = h$ in Ω and $e = 0$ on $\partial\Omega$ such that $e \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. This result is called Calderon–Zygmund estimate and also involves smoothness on the boundary $\partial\Omega$, namely, $\partial\Omega \in C^2$. The interested reader can find additional information in (Dupaigne (2011), p. 252) and (Agmon et al., 1959).

Remark 3.2. Following Remark 3.1 and (Cazenave and Haraux, 1998), if we consider in (4) the initial data e_0 in $H^2(\Omega) \cap H_0^1(\Omega)$ and $F \equiv 0$, then $e \in C([0,\infty), H^2(\Omega))$. On the other hand, if F is Lipschitz continuous on bounded subset of $L^2(\Omega)$ and $e_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the solution e of (4) belongs to

$$C([0,T], D(A)) \cap C^1([0,T]; L^2(\Omega)).$$

3.2 Observer design

In order to prove a stability condition associated to the solution of (3), we start analyzing the following eigenvalue problem for the Laplace operator with a discontinuous potential

$$\begin{cases} -Av = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (11)$$

where A was defined in (5).

The following result shows that C_0 semigroup $S(t)$ associated to A satisfies a stability condition in the sense of Trotter–Kato (see (Trotter, 1958)).

Lemma 3. Let λ_1 be the first eigenvalue of $-A$ in $H_0^1(\Omega)$ such that $\lambda_1 > 0$. Then

$$\|S(t)\|_{\mathcal{L}(L^2)} \leq e^{-\lambda_1 t}, \quad (12)$$

for all $t \geq 0$.

Proof. Let $e_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $e(t) = S(t)e_0$ and consider $g(t) = (e^{\lambda_1 t}\|e(t)\|)^2$, for all $t \geq 0$. Then, integrating by parts we have

$$\begin{aligned} e^{-2\lambda_1 t} g_t(t) &= 2\lambda_1 \int_\Omega e^2(t) dx + 2 \int_\Omega e(t) e_t(t) dx \\ &= 2\lambda_1 \int_\Omega e^2(t) dx + 2 \int_\Omega e(t)[\Delta e(t) - k_0 e(t) 1_{\mathcal{O}}] dx \\ &= 2\lambda_1 \int_\Omega e^2(t) dx - 2 \int_\Omega |\nabla e(t)|^2 dx \\ &\quad - 2k_0 \int_{\mathcal{O}} e^2(t) dx. \end{aligned}$$

Since λ_1 is given by (9), we deduce that $g_t \leq 0$, thus g is decreasing for all $t \geq 0$, which implies $\|S(t)e_0\| \leq e^{-\lambda_1 t}\|e_0\|$ for all $e_0 \in D(A)$. Finally, using density arguments we get (12). This completes the proof of Lemma 3.

We are ready to establish the first result of this paper.

Theorem 4. Let us assume that f is globally Lipschitz with Lipschitz constant L, $f(0) = 0$ and that $-\lambda_1 < -L$, where λ_1 is the minimum eigenvalue of the operator $-A$ defined in (5). Then, the system (2) is an observer for system (1) with sensor location at $\mathcal{O}$.

Remark 3.3. It is clear that the size of λ_1 depends on the choice of k_0 so that the inequality $-\lambda_1 < -L$ can be enforced by selecting k_0, and the gain of the exponential decay depends on its selection.

Proof. For the proof let us assume that $f \in C^1(\mathbb{R})$. We define

$$p(s) = \begin{cases} \dfrac{f(s)}{s} & \text{if } s \neq 0 \\ f'(s) & \text{if } s = 0. \end{cases}$$

We consider a linearization of (4): for $q \in L^2(Q)$ we consider

$$\begin{cases} e_t(t) = Ae(t) + p(q)e, & \forall t \in [0,T], \\ e(\cdot, 0) = e_0(\cdot), \end{cases} \quad (13)$$

Using that f is L–Lipschitz continuous and Lemma 3, we get that

$$\|e(t)\| \leq e^{-\lambda_1 t}\Big(\|e_0\| + e^{-\lambda_1 t} L \int_0^t \|S(-\tau)\| \|e(\tau)\| d\tau \Big)$$

and using Gronwall's inequality it follows that for any $q \in L^2(Q)$, $\|e(t)\| \leq \|e_0\| e^{(-\lambda_1 + L)t}$, in particular for $q = e$, which implies that for $-\lambda_1 < -L$ the observer convergence in $L^2(\Omega)$ is ensured. The general case f L–Lipschitz holds by a standard regularization argument. □

4. ORTHOGONAL MEASUREMENT TO THE AXES

In this section we will consider as domain a rectangle in $\mathbb{R}^2$ and the observation performed at a line orthogonal to one of the sides of the rectangle. Here we lost the regularity of the domain, but the existence and regularity of solutions can be proved and, moreover, we give explicitly the solution.

The framework of this section is based in the works (Schumacher, 1983; Curtain, 1982) in which the compensators design for parabolic system is studied. Our purpose is to make an observer design in a rectangular domain $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$. Define $Q = \Omega \times (0, \infty)$, $\Sigma = \partial\Omega \times$

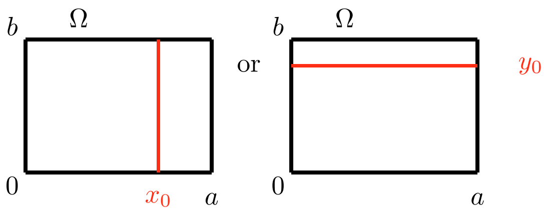

Fig. 1. Rectangular domains with orthogonal measurement to the axes.

$(0,\infty)$. Let x_0 be a fixed point in $(0,a)$. In this section the observations are given by $m(z) = z(x_0,y,t)$. That is, in this section we will consider the linear system:

$$\begin{cases} z_t - \Delta z = 0 & \text{in } Q, \\ z = u & \text{on } \Sigma, \\ z(\cdot,0) = z_0(\cdot) & \text{in } \Omega, \\ m(z) = z(x_0,y,t) & (y,t) \in (0,b)\times(0,\infty). \end{cases} \tag{14}$$

We define:

$$G_M(w,z) = \sum_{k=1}^{M} h_k \sin\Big(\frac{k\pi x}{a}\Big)(w(x_0,y,t) - z(x_0,y,t)),$$

where the constants h_k have to be determined later. Thus, we propose the observer as

$$\begin{cases} w_t - \Delta w = G_M(w,z) & \text{in } Q, \\ w = u & \text{on } \Sigma, \\ w(\cdot,0) = w_0(\cdot) & \text{in } \Omega. \end{cases} \tag{15}$$

This analysis can also be found in the literature as *modal observer design* see, e.g. Vande-Wouwer and Zeitz (2009).

The error system is given by

$$\begin{cases} e_t - \Delta e = \displaystyle\sum_{k=1}^{M} h_k \sin\Big(\frac{k\pi x}{a}\Big) e(x_0,y,t) & \text{in } Q, \\ e = 0 & \text{on } \Sigma, \\ e(\cdot,0) = e_0(\cdot) & \text{in } \Omega, \end{cases} \tag{16}$$

where $e(x,y,t) := w(x,y,t) - z(x,y,t)$, for every $(x,y,t) \in Q$.

We recall the following definition:

Definition 4.1. Let $\mathcal{L}$ be a linear (unbounded) differential operator with eigenvalues $(\lambda_n)_{n\geq 1}$ and associated eigenfunctions $(\varphi_n)_{n\geq 1}$ defined in a set $\Omega \subset \mathbb{R}^N$. We say that $a \in \Omega$ is a *strategic point* of $\mathcal{L}$, if λ_n is simple and $\varphi_n(a) \neq 0$ for every $n \geq 1$.

The eigenvalues $\{\lambda_{m,n}\}_{m,n\in\mathbb{N}}$ for the Laplacian operator with homogeneous boundary in the rectangle can be obtained explicitly: we denote the eigenfunctions by $\varphi_{m,n}(x,y) = \psi_{a,m}(x)\psi_{b,n}(y)$, with

$$\lambda_{m,n} = \Big(\frac{m\pi}{a}\Big)^2 + \Big(\frac{n\pi}{b}\Big)^2$$

and

$$\psi_{a,m}(x) = \sin\Big(\frac{m\pi x}{a}\Big),\ \psi_{b,n}(y) = \sin\Big(\frac{n\pi y}{b}\Big).$$

The second result in this paper is the following.

Theorem 5. Suppose that x_0 is a strategic point of the Dirichlet Laplacian in the interval $(0,a)$. Then, there exist constants $h_i, i = 1,\dots,M$ such that system (15) is an observer for system (1) with sensor location at x_0.

Proof. For the proof, we first suppose that $M = 1$. We take the Fourier series expansion for the error state, that is, $e(x,y,t) = \sum_{k,m=1}^{+\infty} e_{m,n}(t)\varphi_{m,n}(x,y)$, then the main equation of (16) is equivalent to

$$\begin{aligned} &\sum_{m,n=1}^{+\infty} [e'_{m,n}(t) + \lambda_{m,n} e_{m,n}(t)]\varphi_{m,n} \\ &= h_1\psi_{a,1}(x) \sum_{m,n=1}^{+\infty} e_{m,n}(t)\psi_{a,m}(x_0)\psi_{b,n}(y). \end{aligned} \tag{17}$$

Multiplying by $\psi_{a,\ell}(x)$ and integrating on $(0,a)$ we get for $\ell \neq 1$ that

$$\sum_{n=1}^{+\infty} [e'_{\ell,n}(t) + \lambda_{\ell,n} e_{\ell,n}(t)]\psi_{b,n}(y) = 0.$$

Multiplying by $\psi_{b,k}(y)$ and integrating over $(0,b)$, we get a ODE for $e_{\ell,n}(t)$, with solution

$$e_{\ell,n}(t) = e^{-\lambda_{\ell,n}t} e_{\ell,n}(0), \qquad \forall \ell \neq 1. \tag{18}$$

For $\ell = 1$ we get

$$\begin{aligned} &\sum_{n=1}^{+\infty} [e'_{1,n}(t) + (\lambda_{1,n} - h_1\psi_1(x_0))e_{1,n}(t)]\psi_{b,n}(y) = \\ &h_1 \sum_{i\neq 1,\ j=1}^{+\infty} e^{-\lambda_{i,j}t} e_{i,j}(0)\psi_{a,i}(x_0)\psi_{b,j}(y). \end{aligned}$$

So

$$\begin{aligned} e_{1,n}(t) &= e_{1,n}(0)e^{-(\lambda_{1,n} - h_1\psi_{a,1}(x_0))t} \\ &+ \int_0^t e^{-(\lambda_{1,n} - h_1\psi_{a,1}(x_0))(t-\tau)} F_1(x_0,e_0,\tau,h_1)d\tau, \end{aligned}$$

where

$$F_1(x_0,e_0,t,h_1) := \sum_{m\neq 1, n\in\mathbb{N}} e^{-\lambda_{m,n}t} h_1 e_{m,n}(0)\psi_{a,m}(x_0)\varphi_{1,n}.$$

It is clear that we have a gain as soon as $-h_1\psi_{a,1}(x_0) > 0$. The case $M > 1$. For $\ell > M$ we get

$$e_{\ell,n}(t) = e^{-\lambda_{\ell,n}t} e_{\ell,n}(0). \tag{19}$$

Consider

$$X_{M,n} = \begin{pmatrix} e_{1,n} \\ e_{2,n} \\ \vdots \\ e_{M,n} \end{pmatrix}.$$

We get

$$\begin{aligned} X'_{M,n} &= \begin{pmatrix} -\lambda_{1,n} & 0 \\ \cdots & \cdots \\ 0 & -\lambda_{M,n} \end{pmatrix} X_{M,n} \\ &+ \begin{pmatrix} h_1\psi_1(x_0) & \cdots & h_1\psi_M(x_0) \\ \cdots & \cdots & \cdots \\ h_M\psi_1(x_0) & \cdots & h_M\psi_M(x_0) \end{pmatrix} X_{M,n}. \end{aligned}$$

So we have to choose h_i, $i = 1,\dots,M$ such that the matrix

$$\begin{pmatrix} h_1\psi_1(x_0) & \cdots & h_1\psi_M(x_0) \\ \cdots & \cdots & \cdots \\ h_M\psi_1(x_0) & \cdots & h_M\psi_M(x_0) \end{pmatrix}$$

is Hurwitz. □

Remark 4.1. It is clear that a similar analysis can be performed if the observations are taken in the horizontal line $y = y_0$.

5. NUMERICAL EXPERIMENTS

In this section, the designed observer is illustrate for the cases of local internal measurement (Section 3) and orthogonal measurement to the axes (Section 4) for parabolic systems in 2D. For the numerical experiments, $\mathbb{P}_2$–type finite elements and a BDF method are considered for the discretization in space and time respectively. The implementation is carried out in FreeFem++ and Matlab for linear cases and uses the following data: we fix $\Omega = (0,1)^2$, $T = 1$, the observation set $\mathcal{O}$ is a ball centered at $(0.2, 0, 2)$ and radius $R = 0.1$, the time step size is $\Delta t = 5 \times 10^{-2}$. We also consider the initial distributions $z_0 = \cos(2\pi x)$ and $w_0 = 100xy$ for the parabolic system (1) and (2), respectively. External forces $F_z = x(1-x)$, $F_w = y(1-y)$ into (1) and (2) are added, the correction constant is $k_0 = 100$ and by simplicity, controls $u \equiv 10$ are also considered. Respect to the orthogonal measurement to the axes, we consider $x_0 = 0.5$, $M = 1, h_1 = 50$.

In order to appreciate a major numerical visualization, we show cuts of the states z, w and e with the plane $x = 0.2$ in the following figures. More precisely, Figure 2 displays the temperature system and observer designs (2) and (15) without injection, that is, when $m(z) = 0$ for every case.

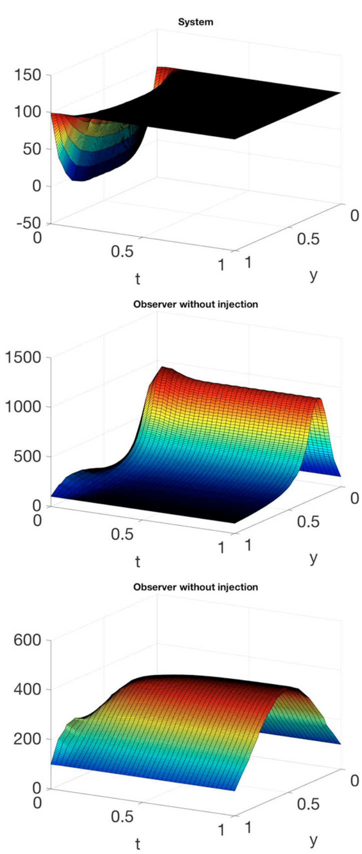

Fig. 2. Cut section of the states z and w by the plane $x = 0.2$. System (top) and observer response without measurements in–domain (middle) or without orthogonal measurement to the axes (bottom).

Once that the local measurement or the orthogonal observation is applied into the observer, we can appreciate the error dynamics and the profiles for the observer designs. Thus, Figure 3 corresponds to the observers with injection $m(z)$ meanwhile that Figure 4 is associated to the error systems for every case.

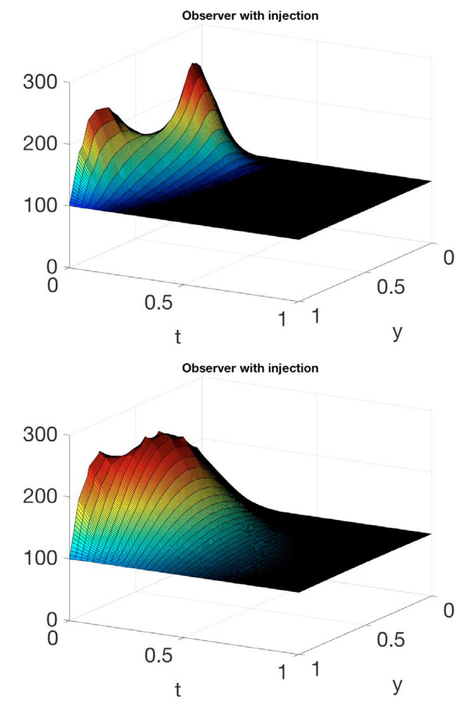

Fig. 3. Observer profiles. Local in–domain measurements (top) and orthogonal measurements to the axes (bottom).

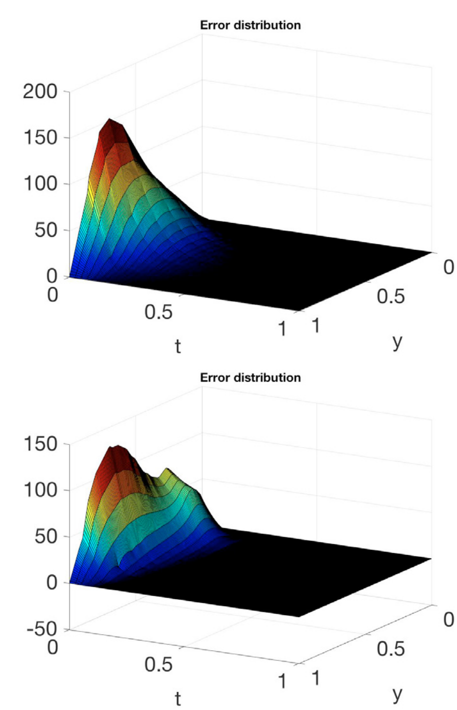

Fig. 4. Error profiles. Local in–domain measurements (top) and orthogonal measurements to the axes (bottom).

6. CONCLUSION

We have shown in this paper that it is possible to construct a globally and exponentially convergent observer for an N-dimensional semilinear parabolic system with arbitrary $N > 2$ when the state is measured on an open subset of a regular domain.

For a linear parabolic system defined on a rectangular domain in $\mathbb{R}^2$ we have shown that a convergent observer can be designed using the measurement of the state along a line parallel to the axes of the rectangle. Although the technique used in this paper do not allow to treat the semilinear case when the observations are performed in a line parallel to the axes in the rectangular case, it is not difficult to extend the results in the linear case for the rectangular domain to hypercubes and to observations on hyperplanes parallel to the sides of the hypercube. It remains an open question the observation on a general curve in the two dimensional case and in N-dimensional domains.

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