Research Article

Cristhian Montoya*

Inverse source problems for the Korteweg–de Vries–Burgers equation with mixed boundary conditions

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Abstract: In this paper, we prove Lipschitz stability results for the inverse source problem of determining the spatially varying factor in a source term in the Korteweg–de Vries–Burgers (KdVB) equation with mixed boundary conditions. More precisely, the Lipschitz stability property is obtained using observation data on an arbitrary fixed sub-domain over a time interval. Secondly, we show that stability property can also be achieved from boundary measurements. Our proofs relies on Carleman inequalities and the Bukhgeim–Klibanov method.

Keywords: Korteweg–de Vries–Burgers equation, inverse source problem, Lipschitz stability, Carleman estimates, Bukhgeim–Klibanov method

MSC 2010: 35R30, 35Q53

1 Introduction

From a physical point of view, the Korteweg–de Vries (KdV) equation describes the propagation of long water waves in channels of shallow depth, in which two phenomena are involved, dispersion (third-order term) and nonlinear convection (nonlinear term). The interaction of these terms gives rise to a wave traveling at constant speed without losing its sharp, called soliton, [25, 27]. In some physical context such as propagation of undular bores in shallow water [21], fluids containing gas bubbles [33], plasma waves [16], nonlinear circuit theory and turbulence [13], a smoothing effect is also added into the model and produces a third phenomenon, dissipation (second-order term). The resulting equation is called the Korteweg–de Vries–Burgers (KdVB) equation [5, 25]. In our case, we consider the KdVB equation on a bounded domain with mixed boundary conditions, namely,

$$\begin{cases} y_t + y_{xxx} - y_{xx} + yy_x = F(x,t) & \text{in } (0,L)\times(0,T), \\ y(0,t) = y(L,t) = 0 & \text{in } (0,T), \\ y_x(L,t) = y_x(0,t) & \text{on } (0,T), \\ y(\cdot\,,0) = 0 & \text{in } (0,L), \end{cases} \tag{1.1}$$

where $y = y(x,t)$ represents the surface elevation of the water wave of quiescent depth at time $(0,T)$ and space $(0,L)$ and $F(x,t)$ is an internal force acting on water wave, which is interpreted as the spatial derivative of the bottom topography of the channel. In this paper, we investigate inverse problems in which we

***Corresponding author: Cristhian Montoya,** Departamento de Matemática, Universidad Técnica Federico Santa Maria, Casilla 110–V, Valparaiso, Chile, e-mail: cristhian.montoya@usm.cl

pretend to determine a spatial varying of bottom topography from either interior measurements or boundary observations.

Assumptions on the source term F as well as well-posedness results concerning (1.1) will be specified later. The main focus of this paper are theoretical stability results for the following inverse source problems.

Inverse source problems. Let $\omega \subset\subset (0, L)$ be a given nonempty sub-domain and $0 < t_0 < T$. Assume the source term $F(x, t) = f(x)R(x, t)$ in (1.1), where R is fixed and known, and let y satisfy (1.1).

Problem 1.1. Determine $f(x)$ from interior observation data $\{y, y_t\}|_{\omega\times(0,T)}$ or from boundary measurements $\{y, y_t\}|_{\{0,L\}\times(0,T)}$. Here, $\{0, L\} \times (0, T)$ is equivalent to $\{(x, t) : x = 0,\ t \in (0, T)\} \cup \{(x, t) : x = L,\ t \in (0, T)\}$.

Problem 1.2. Determine $f(x)$ from observation data $\{y, y_t, y_{tt}\}|_{\omega\times(0,T)}$ and $\{y(\cdot, t_0)\}|_{[0,L]}$, or from measurements $\{y, y_t, y_{tt}\}|_{\{0\}\times(0,T)}$ and $\{y(\cdot, t_0)\}|_{[0,L]}$.

In the previous inverse problems, the source term $f(x)R(x, t)$ is incompletely separated into its spatial and temporal components, and we tackle the problem of determining the spatial component f. In presence of a source term whose decomposition is in the form of separation of variables, i.e., R is space-independent, the term $f(x)R(t)$ can act as the external force modeling a topography which is piecewise constant and has jumps with time varying heights [3].

From a theoretical point of view, some methods for solving inverse source problems have been developed intensively for different kinds of PDEs. Roughly speaking, in elliptic or parabolic equations [9, 11], hyperbolic equations [17, 20, 32], linearized Navier–Stokes system [8] and in a viscoelasticity model [26], the proofs are based on Carleman estimates. A spectral approach for obtaining a source reconstruction formula in hyperbolic equations [34], parabolic equations [15] and in the Stokes system [14] is carried out by a control method and Volterra equations. A new strategy called mixed formulation is presented in [10, 28] and applied to inverse source problems for linear hyperbolic and parabolic equations.

Inverse problems of recovering coefficients for KdV-type equations with Dirichlet and Neumann conditions have been treated by means of optimal control tools and whose measurements are from the final state data [30] or from boundary data [31]. Meanwhile, the determination of the principal coefficient is proved in [2] by the Bukhgeim–Klibanov and Klibanov–Malinsky methods [4, 24], which are based on Carleman estimates for the linearized system. Finally, we invite to the interested reader to see [18, 22, 23, 29] and references therein for a complete description on inverse problems.

To the best of our knowledge, [3] is the only article which studies the inverse problem of retrieving the external source in the KdVB equations with Dirichlet and Neumann boundary conditions. In [3], the authors have used optimization techniques for proving the inverse source problem of recovering the time-varying bottom topography, and where the spatial components f are Dirac measures. Thus the inverse source problem for the KdVB equation with mixed boundary conditions has not been studied so far. The purpose of this paper is to present the first stability and uniqueness theorems to inverse source problems such as Problem 1.1 and Problem 1.2 through the Bukhgeim–Klibanov method. Our proofs are based on Carleman estimates, which are different from the one obtained in [2, 6] for the KdV equation with Neumann boundary conditions and [7] for KdVB equation with mixed boundary conditions. To be more specific, the difference with respect to [2, 6] relies on the weight functions and boundary conditions. In contrast to [7], their Carleman estimate contains weight functions, which are only time dependent and have just one local term on $\omega \times (0, T)$. Indeed, it is not possible to solve our Problem 1.1 using such an inequality; essentially, the weight functions appearing are not the same in each term of the Carleman inequality, and therefore the Bukhgeim–Klibanov method could not be applied.

Let us denote by y and $\tilde{y}$ the solutions to (1.1) for sources $F_1(x, t) = R(x, t)f_1(x)$ and $F_2(x, t) = R(x, t)f_2(x)$, respectively. Additionally, in order to present our main results, we define some sets and impose several assumptions.

Define the set

$$\mathcal{S} = \{F \in C([0, T]; H^3(0, L)) : F_t \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L)),\\ F_{tt} \in L^2(0, T; H^{-1}(0, L))\},$$

and, for any $s \geq 0$, let us define the space

$$Y^s := C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L)).$$

(H1) The input data $(y, \tilde{y})|_{\omega\times(0,T)}$ and $(y, \tilde{y})|_{[0,L]\times\{t_0\}}$ are sufficiently smooth and bounded, i.e., there exists a positive constant M_1 such that

$$\max\{\|y\|_{W^{2,\infty}(0,T;W^{1,\infty}(0,L))}, \|\tilde{y}\|_{W^{2,\infty}(0,T;W^{1,\infty}(0,L))}\} \leq M_1. \tag{1.2}$$

(H2) Assume the existence of positive constants r_0 and M_2. Consider the set

$$\mathcal{M}(R, r_0, M_2) := \{F \in \mathcal{S} : R \in H^1(0,T; L^\infty(0,L)),\ \|R\|_{H^1(0,T;L^\infty(0,L))} \leq M_2,\ |R(x,0)| \geq r_0 \\ \text{and } |F(x,0)| = F(L-x,0) \text{ for all } x \in [0,L]\}.$$

(H3) Let $0 < t_0 < T$, and let $R(x,t)$ be a given fixed function satisfying

$$R(\cdot, t_0) \in C^1([0,L]), \quad \partial_t^k R \in L^\infty((0,L)\times(0,T)), \quad k = 0,1,2.$$

Our first main result is given in the following theorem.

Theorem 1.3. *Let* (H1) *and* (H2) *be satisfied.*
(I) *Then there exists a positive constant $C = C(\omega, M_1, M_2, R, r_0, L, T)$ such that*

$$\|f_1 - f_2\|_{L^2(0,L)} \leq C\|y - \tilde{y}\|_{H^1(0,T;H^2(\omega))}.$$

(II) *Then there exists a positive constant $C = C(M_1, M_2, R, r_0, L, T)$ such that*

$$\begin{aligned}\|f_1 - f_2\|_{L^2(0,L)} \leq C\big(&\|y_x(L,t) - \tilde{y}_x(L,t)\|_{H^1(0,T)} \\ &+ \|y_{xx}(L,t) - \tilde{y}_{xx}(L,t)\|_{H^1(0,T)} \\ &+ \|y_{xx}(0,t) - \tilde{y}_{xx}(0,t)\|_{H^1(0,T)}\big).\end{aligned}$$

Remark 1.4. Theorem 1.3 holds true if the space $W^{2,\infty}(0,T; W^{1,\infty}(0,L))$ on hypothesis (H1) is changed by $W^{1,\infty}(0,T; W^{1,\infty}(0,L))$. However, inequality (1.2) is required in order to solve Problem 1.2.

Remark 1.5. Concerning (H2), a similar condition is assumed in [2] to study an inverse coefficient problem in the KdV-type equation from boundary measurements. Indeed, our spatial anti-symmetry hypothesis on the source F is a consequence of extending the solution of the KdVB equation to negative times. To omit such an assumption will imply to take observations of the solution in a given time $t_0 > 0$; see Theorem 1.6.

Our second main result is given in the following theorem.

Theorem 1.6. *Let* (H1) *and* (H3) *be satisfied. Then there exist positive constants $C_1 = C_1(L, T, t_0, M_1, \omega)$ or $C_2 = C_2(L, T, t_0, M_1)$ such that, for every $y, \tilde{y}$ satisfying* (1.1) *for sources $F_1 = Rf_1$ and $F_2 = Rf_2$, respectively,*

$$\|f_1 - f_2\|_{L^2(0,L)} \leq C_1\big(\|y - \tilde{y}\|_{H^2(0,T;H^2(\omega))} + \|y(\cdot, t_0) - \tilde{y}(\cdot, t_0)\|_{H^3(0,L)}\big)$$

or

$$\begin{aligned}\|f_1 - f_2\|_{L^2(0,L)} \leq C_2\big(&\|y_x(0,\cdot) - \tilde{y}(0,\cdot)\|_{H^2(0,T)} \\ &+ \|y_{xx}(0,\cdot) - \tilde{y}_{xx}(0,\cdot)\|_{H^2(0,T)} \\ &+ \|y(\cdot, t_0) - \tilde{y}(\cdot, t_0)\|_{H^3(0,L)}\big),\end{aligned}$$

where $y, \tilde{y} \in Y^9$, $y_t, \tilde{y}_t \in Y^6$ and $y_{tt}, \tilde{y}_{tt} \in Y^3$,

The rest of this article is organized as follows. In Section 2, we prove regularity results to the KdVB equation considered. In Section 3, one-parameter Carleman estimates for the linearized KdVB equation with mixed boundary conditions are established. Section 4 is dedicated to prove Problem 1.1 (see Theorem 1.3) and Problem 1.2 (see Theorem 1.6) using the Bukhgeim–Klibanov method.

2 Preliminary results

In this section, we establish some regularity properties for the KdVB equation (1.1), which are required in order to prove our inverse problems. The reader interested in studying the well-posedness for the KdVB equation on a bounded domain with different types of boundary conditions could review [1, 7, 19] and references therein.

2.1 Linear problem

We consider the linear problem

$$\begin{cases} y_t + y_{xxx} - y_{xx} = F(x,t) & \text{in } (0,L)\times(0,T), \\ y(0,t) = y(L,t) = 0 & \text{in } (0,T), \\ y_x(L,t) = y_x(0,t) & \text{on } (0,T), \\ y(\cdot,0) = y^0(\cdot) & \text{in } (0,L). \end{cases} \tag{2.1}$$

Recently, Cerpa, Montoya and Zhang [7] proved a well-posedness result for a problem-type (2.1) in the space Y^s for $s \in [0,3]$. We mention their result in the following lemma.

Lemma 2.1. *For every $s \in [0,3]$, let $(F, y_0) \in L^2(0,T;H^{s-1}(0,L)) \times H^s(0,L))$. Then (2.1) admits a unique solution y in the space Y^s.*

Furthermore, there exists a positive constant C such that

$$\|y\|_{Y^s} \leq C\big(\|F\|_{L^2(0,T;H^{s-1}(0,L))} + \|y^0\|_{H^s(0,L)}\big).$$

The proof of Lemma 2.1 is based in semigroup theory and energy estimates for the Cauchy problem

$$y_t + Ay = F, \quad y(0) = y^0,$$

where the operator A is defined by $Au := -u_{xxx} + u_{xx}$, and $D(A) = \{u \in H^3(0,L) \cap H_0^1(0,L) : u_x(0) = u_x(L)\}$.

Now Lemma 2.1 will be used to obtain more regular solutions of (2.1).

Proposition 2.2. *Assume $y^0 \in H^6(0,L) \cap D(A)$ and $F \in \mathcal{S}$. Then the linear KdVB equation (2.1) has a unique solution $y \in C([0,T];H^6(0,L))$ with $y_t \in Y^3$ and $y_{tt} \in Y^0$.*

Proof. Let $z := y_{tt}$. From (2.1), it follows that z satisfies the system

$$\begin{cases} z_t + z_{xxx} - z_{xx} = F_{tt} & \text{in } (0,L)\times(0,T), \\ z(0,t) = z(L,t) = 0 & \text{in } (0,T), \\ z_x(L,t) = z_x(0,t) & \text{on } (0,T), \\ z(\cdot,0) = F_t(\cdot,0) + y^0_{txx}(\cdot) - y^0_{txxx}(\cdot) & \text{in } (0,L). \end{cases} \tag{2.2}$$

By estimating $z(\cdot,0)$ in $L^2(0,L)$ and using Lemma 2.1 on (2.2) with $s = 0$, we obtain

$$\begin{aligned} \|z(\cdot,0)\|_{L^2(0,L)} &\leq C\big(\|F_t\|_{C([0,T];L^2(0,L))} + \|y^0_t\|_{H^3(0,L)}\big), \\ \|z\|_{Y^0} &\leq C\big(\|F_{tt}\|_{L^2(0,T;H^{-1}(0,L))} + \|z(\cdot,0)\|_{L^2(0,L)}\big) \\ &\leq C\big(\|F_{tt}\|_{L^2(0,T;H^{-1}(0,L))} + \|F_t\|_{C([0,T];L^2(0,L))} + \|y^0_t\|_{H^3(0,L)}\big). \end{aligned}$$

Now we need to prove $y_t \in C([0,T];H^3(0,L))$. To do this, let $u := y_t$. Analyzing in the same way the system satisfied by u, we get

$$\begin{aligned} \|u\|_{Y^3} &\leq C\big(\|F_t\|_{L^2(0,T;H^2(0,L))} + \|u(\cdot,0)\|_{H^3(0,L)}\big) \\ &\leq C\big(\|F_t\|_{L^2(0,T;H^2(0,L))} + \|F\|_{C([0,T];H^3(0,L))} + \|y^0\|_{H^6(0,L)}\big). \end{aligned}$$

Finally, it is left to prove $y \in C([0,T]; H^6(0,L))$. From the continuous injection

$$H^1(0,T; H^4(0,L)) \hookrightarrow C([0,T]; H^4(0,L)),$$

we see that $y \in C([0,T]; H^4(0,L))$. On the other hand, taking the main equation of (2.1), we have

$$\begin{aligned}\|y_{xxx}\|_{C([0,T];H^2(0,L))} &\leq C(\|F\|_{C([0,T];H^3(0,L))} + \|y_t\|_{C([0,T];H^2(0,L))} + \|y_{xx}\|_{C([0,T];H^2(0,L))})\\ &\leq C(\|F_t\|_{L^2(0,T;H^2(0,L))} + \|F\|_{C([0,T];H^3(0,L))} + \|y^0\|_{H^6(0,L)}).\end{aligned}$$

Repeating the above arguments, we can obtain the inequality

$$\|y_{xxx}\|_{C([0,T];H^3(0,L))} \leq C(\|F\|_{C([0,T];H^3(0,L))\cap H^1(0,T;H^2(0,L))} + \|y^0\|_{H^6(0,L)}).$$

This completes the proof of Proposition 2.2. □

2.2 Nonlinear problem

In order to prove a similar result to Proposition 2.2 for the nonlinear system

$$\begin{cases} y_t + y_{xxx} - y_{xx} + yy_x = F(x,t) & \text{in } (0,L)\times(0,T),\\ y(0,t) = y(L,t) = 0 & \text{in } (0,T),\\ y_x(L,t) = y_x(0,t) & \text{on } (0,T),\\ y(\cdot,0) = y^0(\cdot) & \text{in } (0,L),\end{cases} \tag{2.3}$$

we will use fixed-point arguments as well as small data. A well-posedness result of system (2.3) with $F \equiv 0$ can be found in [7]. More precisely, the authors of [7] have proved the following lemma.

Lemma 2.3. *Assume $s \in [0,3]$. Then, for every $T > 0$, there exists $\delta > 0$ such that, for any $y_0 \in H^s(0,L)$ satisfying $\|y_0\|_{H^s(0,L)} \leq \delta$, the nonlinear system*

$$\begin{cases} y_t + y_{xxx} - y_{xx} + yy_x = 0 & \text{in } (0,L)\times(0,T),\\ y(0,t) = y(L,t) = 0 & \text{in } (0,T),\\ y_x(L,t) = y_x(0,t) & \text{on } (0,T),\\ y(\cdot,0) = y^0(\cdot) & \text{in } (0,L)\end{cases}$$

admits a unique solution y in the space Y^s.

The fixed-point scheme carried out in [7] can be easily adapted to system (2.3) in order to obtain more regular solutions, which are required to solve our inverse problems.

Let us introduce the space

$$\mathcal{A} = \{y \in C([0,T]; H^6(0,L)) : y_t \in Y^3,\ y_{tt} \in Y^0\}.$$

Proposition 2.4. *Assume $y^0 \in H^6(0,L) \cap D(A)$ and $F \in \mathcal{S}$. Then, for every $T > 0$, there exists $\delta > 0$ such that, for any y^0 and F satisfying*

$$\|y^0\|_{H^3(0,L)} + \|F\|_{C([0,T];H^3(0,L))} \leq \delta,$$

the nonlinear system (2.3) has a unique solution y in the space $\mathcal{A}$.

Sketch of the proof. Let us consider the closed ball $B_R \subset \mathcal{A}$, where $R > 0$ is an appropriate radius to determine. We define the map $\Lambda : B_R \subset \mathcal{A} \to \mathcal{A}$ by $\mathcal{A}(v) := y$, and y satisfies the system

$$\begin{cases} y_t + y_{xxx} - y_{xx} = F(x,t) - vv_x & \text{in } (0,L)\times(0,T),\\ y(0,t) = y(L,t) = 0 & \text{in } (0,T),\\ y_x(L,t) = y_x(0,t) & \text{on } (0,T),\\ y(\cdot,0) = y^0(\cdot) & \text{in } (0,L).\end{cases}$$

Using the semigroup $\{S(t)\}_{t\geq 0}$ associated to the operator A as well as Proposition 2.2, there exist positive constants C_1, C_2 and C_3 such that

$$\|\Lambda(v)\|_{\mathcal{A}} \leq C_1\|y^0\|_{H^3(0,L)} + C_2\|v\|_{\mathcal{A}}^2 + C_3\|F\|_{C([0,T];H^3(0,L))}. \tag{2.4}$$

Consider $R > 0$ such that, for any $m_0, n_0 \geq 2$,

$$R = m_0 \max\{C_1, C_3\}(\|y^0\|_{H^3(0,L)} + \|F\|_{C([0,T];H^3(0,L))}) \quad \text{and} \quad C_2 R \leq \frac{1}{2n_0}.$$

Let $\delta = (2m_0 n_0 C_2 \max\{C_1, C_3\})^{-1}$. From (2.4), we get $\|\Lambda(v)\|_{\mathcal{A}} \leq R$. Moreover, for every $v, w \in B_R$, Λ is a contraction mapping on B_R since

$$\|\Lambda(v) - \Lambda(w)\|_{\mathcal{A}} \leq C_4(\|v\|_{\mathcal{A}} + \|w\|_{\mathcal{A}})\|w - v\|_{\mathcal{A}} \leq \frac{1}{n_0}\|v - w\|_{\mathcal{A}}.$$

Therefore, Λ has a unique fixed point $w \in \mathcal{A}$, which is the solution of (2.3). □

Remark 2.5. From the above results and interpolation arguments, it is possible to deduce the existence of a unique solution y of (2.3) in $W^{1,\infty}((0,T); W^{1,\infty}(0,L))$. Recall that this regularity is required for solving Problem 1.1; see Theorem 1.3.

On the other hand, the existence of a unique solution y of (2.3) in the space $W^{2,\infty}((0,T); W^{1,\infty}(0,L))$ (which is necessary for solving Problem 1.2, Theorem 1.6) can be obtained using the structure of the proof of Proposition 2.2 and Proposition 2.4. We only mention the result, and therefore we omit the proof.

Proposition 2.6. *Assume $y^0 \in H^9(0,L) \cap D(A)$ and $F \in Y^6$. Then the nonlinear system* (2.3) *has a unique solution y in the space Y^9. Furthermore, $y_t \in Y^6$, $y_{tt} \in Y^3$, $y_{ttt} \in Y^0$ with*

$$F_t \in Y^3, \quad F_{tt} \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L)) \quad \textit{and} \quad F_{ttt} \in L^2(0,T; H^{-1}(0,L)),$$

respectively.

3 Carleman inequalities

In this section, two Carleman estimates for the linearized KdVB equation with constant coefficients on the domain defined here by $Q := (0,L) \times (-T,T)$ will be proved. As mentioned in the introduction, few authors have proved Carleman inequalities for the KdV and KdVB equations posed on a bounded interval (see [2, 6] and [7], respectively). In this article, we tackle with mixed boundary conditions like in [7]. In order to apply the Bukhgeim–Klibanov method for solving inverse problems, it is crucial to prove a one-parameter Carleman estimate with the same exponential weight function in each integral term.

Concerning the inverse source problem with boundary observation data (see Theorem 1.3, case (II)), a Carleman inequality with boundary terms on its right-hand side as well as equal exponential weight functions in each term is necessary. It can be solved by adapting the Carleman estimate for the linearized KdV equation proved in [2], which involves two parameters. However, we propose an alternative construction using only one parameter since our inverse problem concerns the source and not the coefficients. In fact, it implies different regularity conditions to the ones presented in [2], which were studied in the previous section.

We also underline that the Carleman inequality given in Proposition 3.1 is different to the one established in [7]. Namely, in this reference, a Carleman estimate containing only time-dependent weight functions and one local term on $\omega \times (0,T)$ is proved. Indeed, it is not possible to solve our inverse source problem using such an inequality; essentially, the exponential weight functions appearing are not the same in each term, and therefore the Bukhgeim–Klibanov method cannot be applied.

Let us consider the operator P given by

$$P := \partial_t + \partial_{xxx} - \partial_{xx} + a(x,t)\partial_x + b(x,t), \tag{3.1}$$

where $a \in L^\infty(0,T; W^{1,\infty}(0,L))$, $b \in L^\infty(Q)$, and the space

$$\mathcal{E} = \{v \in L^2(-T,T; H^3(0,L) \cap H_0^1(0,L)) : Pv \in L^2(Q)\}. \tag{3.2}$$

3.1 Carleman estimate with local terms

Let ω be a nonempty open subset of $(0, L)$, and let $\phi \in C^4([0, L])$ satisfy

$$\phi > 0 \text{ in } [0, L], \quad \phi_x(0) < 0, \quad \phi_x(L) > 0 \tag{3.3}$$

and

$$\phi_{xx} < 0 \text{ in } \overline{(0, L)\backslash\omega}, \qquad \min_{x\in\bar{\omega}=[\ell_1,\ell_2]} \phi(x) = \phi(\ell_2), \qquad \max_{x\in[0,L]} \phi(x) = \phi(L). \tag{3.4}$$

Moreover, we consider the weight functions

$$\alpha(x, t) := \phi(x)\xi(t), \quad \xi(t) := \frac{1}{(T + t)(T - t)}. \tag{3.5}$$

An example of the weight function ϕ is the following: given $\omega = (\ell_1, \ell_2)$, we have

$$\phi(x) = \begin{cases} \varepsilon x^3 - 3\ell_1 x^2 - x + C_1 & \text{in } [0, \ell_1], \\ -\varepsilon x^3 + (1 + 3\varepsilon L^2)x + C_2 & \text{in } [\ell_2, L], \end{cases}$$

where $C_1 = 2\varepsilon L^3 + L + C_2$, $0 < \varepsilon < 1$ and $C_2 \gg 1$.

The first of our main results in this section is the following Carleman estimate.

Proposition 3.1. *Let ϕ and ξ be defined by* (3.5). *There exist $s_0 > 0$ and a positive constant C depending on L, ω, T, $\|a\|_{L^\infty(0,T;W^{1,\infty}(0,L))}$, $\|b\|_{L^\infty(Q)}$ and $\|\phi\|_{C^4[0,L]}$ such that, for every $s \geq s_0$, for all $y \in \mathcal{E}$ defined by* (3.2), *we have*

$$\begin{aligned}
&\iint_Q [s^5\xi^5|y|^2 + s^3\xi^3|y_x|^2 + s\xi|y_{xx}|^2]e^{-2s\alpha}\,dx\,dt \\
&\qquad + s\int_{-T}^{T} \xi|y_{xx}(0, t)|^2 e^{-2s\alpha(0,t)} + \xi|y_{xx}(L, t)|^2 e^{-2s\alpha(L,t)}\,dt \\
&\leq C\Bigg(\iint_Q |Py|^2 e^{-2s\alpha}\,dx\,dt + \iint_{\omega\times(-T,T)} [(s\xi)^5|y|^2 + s^3\xi^3|y_x|^2 + s\xi|y_{xx}|^2]e^{-2s\alpha}\,dx\,dt\Bigg).
\end{aligned} \tag{3.6}$$

Proof of Proposition 3.1. The proof is divided in three steps: The first one is a classical setting; both decomposition and change to an appropriate operator involving the weight functions (3.3)–(3.5) is considered. In the second one, we will estimate global integrals using integration by parts. Finally, we will return to the principal variable y for obtaining the desired inequality (3.6).

Step 1: Change of variable. In this step, we consider the differential operator satisfied by a new variable z, which will be y up to a weight function. More precisely, for every $y \in \mathcal{E}$ and $s > 0$, we set $z = e^{-s\alpha}y$, and we denote by L_1 and L_2 the skew-adjoint and self-adjoint parts of the operator P defined in (3.1), respectively. Thus, if $F_s = e^{-s\alpha}F$, the identity $e^{-s\alpha}P(e^{s\alpha}z) = F_s$ is equivalently to

$$L_1 z + L_2 z = F_s + R_s,$$

where

$$\begin{aligned}
L_1 z &:= z_t + z_{xxx} + 3s^2(\alpha_x)^2 z_x, \\
L_2 z &:= 3s(\alpha_x)z_{xx} + s^3(\alpha_x)^3 z + 3s(\alpha_{xx})z_x
\end{aligned}$$

and

$$\begin{aligned}
R_s := {} & s^2(\alpha_x)^2 z + s\alpha_{xx}z + 2s\alpha_x z_x + z_{xx} \\
& - [3s^2\alpha_x\alpha_{xx} + s\alpha_{xxx} + (1 + a(x, t))s\alpha_x + s\alpha_t]z \\
& - (1 + a(x, t))z_x - b(x, t)z.
\end{aligned}$$

Thus we have

$$\|L_1 z\|^2_{L^2(Q)} + \|L_2 z\|^2_{L^2(Q)} + 2\langle L_1 z, L_2 z\rangle = \|F_s + R_s\|^2_{L^2(Q)},$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(Q)$ inner product. In the next step, we carry out several estimates for the terms that arise of the inner product $\langle L_1 z, L_2 z\rangle$.

Step 2: Estimates of global terms. In this step, we use several times integration by parts in order to develop the nine terms appearing in $\langle L_1 z, L_2 z\rangle$. Then

$$\begin{aligned}
I^{1,1} &:= \langle L_1^1 z, L_2^1 z\rangle = 3s \iint_Q \alpha_x z_t z_{xx}\,dx\,dt \\
&= -3s \iint_Q \alpha_{xx} z_t z_x\,dx\,dt + \frac{3s}{2}\iint_Q \alpha_{xt}|z_x|^2\,dx\,dt \\
&= -3s \iint_Q \xi\phi_{xx} z_t z_x\,dx\,dt + \frac{3s}{2}\iint_Q \xi_t\phi_x|z_x|^2\,dx\,dt, \\
I^{1,2} &:= \langle L_1^1 z, L_2^2 z\rangle = s^3 \iint_Q (\alpha_x)^3 z_t z\,dx\,dt = -\frac{3s^3}{2}\iint_Q \xi^2\xi_t\phi_x^3|z|^2\,dx\,dt, \\
I^{1,3} &:= \langle L_1^1 z, L_2^3 z\rangle = 3s \iint_Q \alpha_{xx} z_x z_t\,dx\,dt = \frac{3s}{2}\iint_Q \alpha_{xt}|z_x|^2\,dx\,dt - I^{1,1} \\
&= \frac{3s}{2}\iint_Q \xi_t\phi_x|z_x|^2\,dx\,dt - I^{1,1}, \qquad (3.7) \\
I^{2,1} &:= \langle L_1^2 z, L_2^1 z\rangle = 3s \iint_Q \alpha_x z_{xx} z_{xxx}\,dx\,dt \\
&= -\frac{3s}{2}\iint_Q \xi\phi_{xx}|z_{xx}|^2\,dx\,dt + \underbrace{\frac{3s}{2}\int_{-T}^{T} (\xi\phi_x|z_{xx}|^2\big|_{x=0}^{x=L})\,dt}_{A}. \qquad (3.8)
\end{aligned}$$

From (3.3), $A \geq 0$. In fact, there exists a positive constant $C = C(L, \|\phi_x\|_{L^\infty(0,L)})$ such that

$$A \geq Cs \int_{-T}^{T} \xi(|z_{xx}(0,t)|^2 + |z_{xx}(L,t)|^2)\,dt.$$

We continue with the estimates and have

$$\begin{aligned}
I^{2,2} &:= \langle L_1^2 z, L_2^2 z\rangle = s^3 \iint_Q (\alpha_x)^3 z z_{xxx}\,dx\,dt \\
&= -3s^3 \iint_Q (\alpha_x)^2\alpha_{xx} z z_{xx}\,dx\,dt - s^3\iint_Q (\alpha_x)^3 z_x z_{xx}\,dx\,dt \\
&= 3s^3 \iint_Q [(\alpha_x)^2\alpha_{xx}]_x z z_x\,dx\,dt + 3s^3\iint_Q (\alpha_x)^2\alpha_{xx}|z_x|^2\,dx\,dt \\
&\quad + \frac{s^3}{2}\iint_Q [(\alpha_x)^3]_x|z_x|^2\,dx\,dt - \underbrace{\frac{s^3}{2}\int_{-T}^{T} ((\alpha_x)^3|z_x|^2\big|_{x=0}^{x=L})\,dt}_{B} \\
&= \frac{9s^3}{2}\iint_Q (\alpha_x)^2\alpha_{xx}|z_x|^2\,dx\,dt - \frac{3s^3}{2}\iint_Q [(\alpha_x)^2\alpha_{xx}]_{xx}|z|^2\,dx\,dt - B, \qquad (3.9) \\
I^{2,3} &:= \langle L_1^2 z, L_2^3 z\rangle = 3s \iint_Q \alpha_{xx} z_x z_{xxx}\,dx\,dt \\
&= -3s \iint_Q \alpha_{xxx} z_x z_{xx}\,dx\,dt - 3s\iint_Q \alpha_{xx}|z_{xx}|^2\,dx\,dt + 3s\int_{-T}^{T} (\alpha_{xx} z_x z_{xx}\big|_{x=0}^{x=L})\,dt \\
&= -3s \iint_Q \alpha_{xx}|z_{xx}|^2\,dx\,dt + \frac{3s}{2}\iint_Q \alpha_{xxxx}|z_x|^2\,dx\,dt + \tilde{C}
\end{aligned}$$

$$= -3s \iint_Q \xi \phi_{xx} |z_{xx}|^2 \, dx \, dt + \frac{3s}{2} \iint_Q \xi \phi_{xxxx} |z_x|^2 \, dx \, dt + \tilde{C}, \tag{3.10}$$

where

$$\tilde{C} := 3s \int_{-T}^{T} (\xi \phi_{xx} z_x z_{xx} |_{x=0}^{x=L}) \, dt - \frac{3s}{2} \int_{-T}^{T} (\xi \phi_{xxx} |z_x|^2 |_{x=0}^{x=L}) \, dt. \tag{3.11}$$

The last estimates are obtained as follows:

$$\begin{aligned} I^{3,1} := \langle L_1^3 z, L_2^1 z \rangle &= 9s^3 \iint_Q (\alpha_x)^3 z_x z_{xx} \, dx \, dt \\ &= -\frac{9s^3}{2} \iint_Q [(\alpha_x)^3]_x |z_x|^2 \, dx \, dt + \frac{9s^3}{2} \int_{-T}^{T} ((\alpha_x)^3 |z_x|^2 |_{x=0}^{x=L}) \, dt \\ &= -\frac{27s^3}{2} \iint_Q (\alpha_x)^2 \alpha_{xx} |z_x|^2 \, dx \, dt + \underbrace{\frac{9s^3}{2} \int_{-T}^{T} ((\alpha_x)^3 |z_x|^2 |_{x=0}^{x=L}) \, dt}_{D}. \end{aligned} \tag{3.12}$$

Using (3.3), (3.9) and (3.12), we deduce that $D - B = 4B \geq 0$. Thus there exists a positive constant

$$C = C(L, \|\phi_x\|_{L^\infty(0,L)}) \quad \text{such that} \quad D - B \geq Cs^3 \int_{-T}^{T} \xi^3 [|z_x(L,t)|^2 + |z_x(0,t)|^2] \, dt.$$

Besides,

$$\begin{aligned} I^{3,2} := \langle L_1^3 z, L_2^2 z \rangle &= 3s^5 \iint_Q (\alpha_x)^5 z z_x \, dx \, dt \\ &= -\frac{3s^5}{2} \iint_Q [(\alpha_x)^5]_x |z|^2 \, dx \, dt = -\frac{15s^5}{2} \iint_Q \xi^5 (\phi_x)^4 \phi_{xx} |z|^2 \, dx \, dt. \end{aligned}$$

Taking into account that $\phi_{xx} < 0$ in $\overline{(0,L) \backslash \omega}$ (see (3.4)), we can estimate $I^{3,2}$ by

$$Cs^5 \iint_Q \xi^5 |z|^2 \, dx \, dt - Cs^5 \iint_{\omega \times (-T,T)} \xi^5 |z|^2 \, dx \, dt \leq -\frac{15s^5}{2} \iint_Q \xi^5 (\phi_x)^4 \phi_{xx} |z|^2 \, dx \, dt$$

for any $s \geq C(L, \omega, T, \|\phi_{xx}\|_{L^\infty(0,L)})$.

Finally,

$$I^{3,3} := \langle L_1^3 z, L_2^3 z \rangle = 9s^3 \iint_Q (\alpha_x)^2 \alpha_{xx} |z_x|^2 \, dx \, dt = 9s^3 \iint_Q \xi^3 (\phi_x)^2 \phi_{xx} |z_x|^2 \, dx \, dt.$$

Now we denote by $I_1^{2,1}$ and $I_1^{2,3}$ the first terms of (3.8) and (3.10), respectively. Thus, using again condition (3.4), for any $s \geq C(L, \omega, T, \|\phi_{xx}\|_{L^\infty(0,L)})$, we have

$$I_1^{2,1} + I_1^{2,3} = -\frac{9s}{2} \iint_Q \xi \phi_{xx} |z_{xx}|^2 \, dx \, dt \geq Cs \iint_Q \xi |z_{xx}|^2 \, dx \, dt - Cs \iint_{\omega \times (-T,T)} \xi |z_{xx}|^2 \, dx \, dt.$$

If $I_1^{2,2}$ and $I_1^{3,1}$ denote the first terms of $I^{2,2}$ and $I^{3,1}$, respectively, then

$$I_1^{2,2} + I_1^{3,1} + I^{3,3} = 0.$$

Additionally, the second term of $I^{2,3}$, denoted by $I_2^{2,3}$, can be upper bounded (after integrating by parts and using Young's inequality) by

$$I_2^{2,3} \leq Cs^3 \iint_Q \xi^3 |z_x|^2 \, dx \, dt \leq \iint_Q (s^5 \xi^5 |z|^2 + s\xi |z_{xx}|^2) \, dx \, dt \tag{3.13}$$

for any $s \geq C(L, \omega, T, \|\phi\|_{C^4[0,L]})$.

As a consequence of (3.13), the first term in (3.7) as well as the second term in (3.10) can be estimated by the left-hand side of (3.13). By considering the constant $\tilde{C}$ defined in (3.11), at this moment, for any $s \geq C(L, \omega, T, \|\phi\|_{C^4[0,L]})$, we can deduce the inequality

$$\begin{aligned}
\iint_Q [s^5\xi^5|z|^2 + s^3\xi^3|z_x|^2 + s\xi|z_{xx}|^2]\,dx\,dt + s\int_{-T}^{T} \xi(|z_{xx}(0,t)|^2 + |z_{xx}(L,t)|^2)\,dt \\
+ s^3\int_{-T}^{T} \xi^3[|z_x(L,t)|^2 + |z_x(0,t)|^2]\,dt + \tilde{C} \\
\leq C\Big(\|F_s\|^2_{L^2(Q)} + \|R_s\|^2_{L^2(Q)} + \iint_{\omega\times(-T,T)} [s^5\xi^5|z|^2 + s\xi|z_{xx}|^2]\,dx\,dt\Big).
\end{aligned}$$

Finally, we shall estimate $\tilde{C}$. Indeed,

$$\begin{aligned}
|\tilde{C}| = \Big|3s\int_{-T}^{T} \xi[\phi_{xx}(L,t)z_x(L,t)z_{xx}(L,t) - \phi_{xx}(0,t)z_x(0,t)z_{xx}(0,t)]\,dt \\
- \frac{3s}{2}\int_{-T}^{T} \xi[\phi_{xxx}(L,t)|z_x(L,t)|^2 - \phi_{xxx}(0,t)|z_x(0,t)|^2]\,dt\Big| \\
\leq Cs^2\int_{-T}^{T} \xi[|z_x(L,t)|^2 + |z_x(0,t)|^2]\,dt + C\int_{-T}^{T} \xi(|z_{xx}(0,t)|^2 + |z_{xx}(L,t)|^2)\,dt.
\end{aligned}$$

Hence there exists a constant C depending on L, ω, T, $\|\phi\|_{C^4[0,L]}$, $b \in L^\infty(Q)$ and $\|a\|_{L^\infty(0,T;W^{1,\infty}(0,L))}$ such that, for any $s \geq C$,

$$\begin{aligned}
\iint_Q (s^5\xi^5|z|^2 + s^3\xi^3|z_x|^2 + s\xi|z_{xx}|^2)\,dx\,dt + s\int_{-T}^{T} \xi(|z_{xx}(0,t)|^2 + |z_{xx}(L,t)|^2)\,dt \\
+ s^3\int_{-T}^{T} \xi^3(|z_x(L,t)|^2 + |z_x(0,t)|^2)\,dt \\
\leq C\Big(\iint_Q |F|e^{-2s\alpha}\,dx\,dt + \iint_{\omega\times(-T,T)} (s^5\xi^5|z|^2 + s\xi|z_{xx}|^2)\,dx\,dt\Big).
\end{aligned} \tag{3.14}$$

Step 3: Return to the main variable. In this step, we turn back to our original function for obtaining the desired inequality (3.6). Taking into account that our change of variable was given by $y = e^{s\alpha}z$, a direct computation allow us to get

$$\begin{aligned}
|y_x|^2e^{-2s\alpha} &\leq C(s^2\xi^2|z|^2 + |z_x|^2), \\
|y_{xx}|^2e^{-2s\alpha} &\leq C(s^4\xi^4|z|^2 + s^2\xi^2|z_x|^2 + |z_{xx}|^2).
\end{aligned}$$

On the other hand,

$$|z_{xx}|^2 \leq Ce^{-2s\alpha}(s^4\xi^4|y|^2 + s^2\xi^2|y_x|^2 + |y_{xx}|^2)$$

Finally, substituting the previous estimates into (3.14) yields inequality (3.6). This completes the proof of Proposition 3.1. □

3.2 Carleman estimate with boundary terms

In order to prove the Lipschitz stability from boundary measurements, we first proof a global Carleman inequality for the linearized KdVB equation with boundary terms at $x = 0$. The Carleman estimate will be developed using the scheme of the previous subsection.

Let $\psi \in C^4[0, L]$ satisfy

$$\psi > 0 \quad \text{in } [0, L], \tag{3.15}$$

$$\psi_{xx} < 0 \quad \text{in } [0, L], \tag{3.16}$$

$$\psi_x(L) > 0, \quad \psi_x(0) > 0. \tag{3.17}$$

Moreover, we consider the weight functions

$$\beta(x, t) := \psi(x)\xi(t), \quad \xi(t) := \frac{1}{(T + t)(T - t)}. \tag{3.18}$$

It is very easy to verify that, for every $a > 0$, $\psi(x) = ax^3 - 4aLx^2 + 6aL^2x + d$ satisfies (3.16) and (3.17); meanwhile, (3.15) holds for $d \gg 1$.

The second main result of this section is the following Carleman estimate.

Proposition 3.2. *Let β and ξ be defined by* (3.15)–(3.18). *There exist $s_0 > 0$ and a positive constant C depending on L, T, $\|a\|_{L^\infty(0,T;W^{1,\infty}(0,L))}$, $\|b\|_{L^\infty(Q)}$ and $\|\psi\|_{C^4[0,L]}$ such that, for every $s \geq s_0$, for all $y \in \mathcal{E}$ defined by* (3.2), *we have*

$$\begin{aligned}\iint_Q [s^5\xi^5|y|^2 + s^3\xi^3|y_x|^2 + s\xi|y_{xx}|^2]e^{-2s\beta}\,dx\,dt \\ \leq C\Big(\iint_Q |Py|^2 e^{-2s\beta}\,dx\,dt + \int_{-T}^{T} [s^3\xi^3|y_x(0, t)|^2 + s\xi|y_{xx}(0, t)|^2]e^{-2s\beta(0,t)}\,dt\Big).\end{aligned} \tag{3.19}$$

Sketch of proof. We follow closely the proof of Proposition 3.1. Thus, from the change of variable $z = e^{-s\beta}y$ and $F_s = e^{-s\beta}Py$, we have again $\|L_1z\|^2_{L^2(Q)} + \|L_2z\|^2_{L^2(Q)} + 2\langle L_1z, L_2z\rangle = \|F_s + R_s\|^2_{L^2(Q)}$, where L_1, L_2, R_s are defined like in step 1 of the proof of Proposition 3.1, but considering β instead of α. Now, from step 2 of the proof of Proposition 3.1, we can deduce the inequality

$$\begin{aligned}&-\frac{9s}{2}\iint_Q \xi\psi_{xx}|z_{xx}|^2\,dx\,dt - \frac{15s^5}{2}\iint_Q \xi^5(\psi_x)^4\psi_{xx}|z|^2\,dx\,dt \\ &\quad + \frac{3s}{2}\iint_Q \xi\psi_{xxxx}|z_x|^2\,dx\,dt + \frac{3s}{2}\int_{-T}^{T} \xi\psi_x|z_{xx}|^2\Big|_{x=0}^{x=L}\,dt + 4s^3\int_{-T}^{T} (\psi_x\xi)^3|z_x|^2\Big|_{x=0}^{x=L}\,dt \\ &\quad + 3s\int_{-T}^{T} \big(\xi\psi_{xx}z_xz_{xx}\big|_{x=0}^{x=L}\big)\,dt - \frac{3s}{2}\int_{-T}^{T} \big(\xi\psi_{xxx}|z_x|^2\big|_{x=0}^{x=L}\big)\,dt \\ &\leq 2(\|F_s\|^2_{L^2(Q)} + \|R_s\|^2_{L^2(Q)}).\end{aligned}$$

On the other hand, using condition (3.16), we can estimate the third term on the left-hand side of the previous inequality like in (3.13) for all $s \geq C$, where $C = C(L, T, \|\psi\|_{C^4[0,L]})$. Then there exists a positive constant C only depending on L, T, $\|\psi\|_{C^4[0,L]}$ such that, for any $s \geq C$,

$$\begin{aligned}&\iint_Q [s^5\xi^5|z|^2 + s^3\xi^3|z_x|^2 + s\xi|z_{xx}|^2]\,dx\,dt \\ &\quad + \frac{3s}{2}\int_{-T}^{T} \xi(t)\big(\psi_x(L)|z_{xx}(L, t)|^2 - \psi_x(0)|z_{xx}(0, t)|^2\big)\,dt \\ &\quad + 4s^3\int_{-T}^{T} \xi^3(t)\big(\psi_x^3(L)|z_x(L, t)|^2 - \psi_x^3(0)|z_x(0, t)|^2\big)\,dt \\ &\quad + \underbrace{3s\int_{-T}^{T} \big(\xi\psi_{xx}z_xz_{xx}\big|_{x=0}^{x=L}\big)\,dt - \frac{3s}{2}\int_{-T}^{T} \big(\xi\psi_{xxx}|z_x|^2\big|_{x=0}^{x=L}\big)\,dt}_{J(z_x,z_{xx})} \leq C(\|F_s\|^2_{L^2(Q)} + \|R_s\|^2_{L^2(Q)}).\end{aligned} \tag{3.20}$$

Now, using condition (3.17) in (3.20), for any $s \geq C$, we can deduce

$$\iint_Q [s^5\xi^5|z|^2 + s^3\xi^3|z_x|^2 + s\xi|z_{xx}|^2]\,dx\,dt + s\int_{-T}^{T} \xi(t)\psi_x(L)|z_{xx}(L,t)|^2\,dt$$
$$+ s^3\int_{-T}^{T} \xi^3(t)\psi_x^3(L)|z_x(L,t)|^2\,dt + J(z_x, z_{xx})$$
$$\leq C\Big(s^3\int_{-T}^{T} \xi^3(t)|z_x(0,t)|^2\,dt + s\int_{-T}^{T} \xi(t)|z_{xx}(0,t)|^2\,dt + \|F_s\|^2_{L^2(Q)} + \|R_s\|^2_{L^2(Q)}\Big). \tag{3.21}$$

Applying the Cauchy–Schwartz inequality, $J(z_x, z_{xx})$ can be upper estimated by the boundary terms associated to $|z_x(0,\cdot)|$, $|z_{xx}(0,\cdot)|$ in the right-hand side of (3.21) and by the terms $|z_x(L,\cdot)|$, $|z_{xx}(L,\cdot)|$ in the left-hand side of (3.21). This completes the sketch of proof of Proposition 3.2. □

Remark 3.3. A result similar to Proposition 3.2 with boundary terms at $x = L$ can be obtained if the signs of ψ_x given in (3.17) are changed. This new proposition will be equivalent to [2, Theorem 2.2], but involving only one parameter.

4 Inverse problems

This section contains the proof of Lipschitz stability results stated in Theorem 1.3 and Theorem 1.6 concerning inverse problems of recovering the source $F(x,t) = R(x,t)f(x)$ in equation (1.1), either from interior observation data or boundary observation data of the solution. As mention before, the proof relies upon the Bukhgeim–Klibanov method.

Step 1: Auxiliary system. First of all, we consider two external sources $F_1 = Rf_1$, $F_2 = Rf_2$ and the corresponding solutions to (1.1), y and $\tilde{y}$, respectively.

Let us define $u(x,t) := y(x,t) - \tilde{y}(x,t)$ and $G(x,t) = R(x,t)(f_1(x) - f_2(x))$. It follows that u satisfies the system

$$\begin{cases} u_t + u_{xxx} - u_{xx} + \tilde{y}u_x + y_x u = G(x,t) & \text{in } (0,L)\times(0,T),\\ u(0,t) = u(L,t) = 0 & \text{in } (0,T),\\ u_x(L,t) = u_x(0,t) & \text{on } (0,T),\\ u(\cdot,0) = 0 & \text{in } (0,L). \end{cases} \tag{4.1}$$

In order to apply the Bukhgeim–Klibanov method, we consider $z := u_t$, which satisfies

$$\begin{cases} z_t + z_{xxx} - z_{xx} + \tilde{y}z_x + y_x z = \widetilde{G(x,t)} & \text{in } (0,L)\times(0,T),\\ z(0,t) = z(L,t) = 0 & \text{in } (0,T),\\ z_x(L,t) = z_x(0,t) & \text{on } (0,T),\\ z(\cdot,0) = G(\cdot,0) & \text{in } (0,L), \end{cases} \tag{4.2}$$

where $\widetilde{G(x,t)} = G_t(x,t) - \tilde{y}_t u_x - y_{xt}u$.

Observe that f_1, f_2 appears not only in the source term of (4.2), but also in the initial condition. In addition, although the Carleman estimate (3.6) is independent of initial data (see Proposition 3.1), we shall extend system (4.2) to negative times in order to use such a Carleman inequality.

Step 2: Extension to negative time. Now we extend the systems (4.1) in u and (4.2) in z to negative times. To do this, we define the symmetric extension of any function g defined on $[0,L]\times[0,T]$ by

$$g^{ee}(x,t) := \begin{cases} g(x,t) & \text{in } [0,L]\times[0,T],\\ g(L-x,-t) & \text{in } [0,L]\times[-T,0). \end{cases}$$

In a similar way, we define the anti-symmetric extension to $[0, L] \times [-T, T]$ of any function g defined on $[0, L] \times [0, T]$ by

$$g^{ae}(x,t) := \begin{cases} g(x,t) & \text{in } [0,L]\times[0,T], \\ -g(L-x,-t) & \text{in } [0,L]\times[-T,0). \end{cases}$$

One should observe that this extension satisfies $g_1^{ee} g_2^{ee} = (g_1 g_2)^{ee}$. Besides, any idea of extending the solution z of (4.2) to negative times is closely linked to extend the solutions $y, \tilde{y}$ of (1.1) to negative times and assume that the initial data $z(\cdot, 0)$ satisfies $|z(x,0)| = z(L-x,0)$ for all $x \in [0, L]$, that is,

$$|R(x,0)|\,|f_1(x) - f_2(x)| = R(L-x,0)\big(f_1(L-x) - f_2(L-x)\big) \quad \text{for all } x \in [0, L].$$

Thus, without loss of generality, we define the symmetric extension of the solution y (resp. $\tilde{y}$) of (1.1) to negative times. Let $v := z^{ee}$ on $[0, L] \times [-T, T]$. It is easy to verify that v satisfies the system

$$\begin{cases} v_t + v_{xxx} - v_{xx} + \tilde{y}^{ee} v_x + y_x^{ee} v = \widetilde{G^{ae}(x,t)} & \text{in } [0,L]\times[-T,T], \\ v(0,t) = v(L,t) = 0 & \text{in } (0,T), \\ v_x(L,t) = v_x(0,t) & \text{on } (-T,T), \\ v(\cdot,0) = G(\cdot,0) & \text{in } (0,L). \end{cases} \tag{4.3}$$

The main equation of (4.3) is eligible for the Carleman estimates of Proposition 3.1 and Proposition 3.2 with $a(x,t) = \tilde{y}^{ee}$ and $b(x,t) = y_x^{ee}$. Moreover, the regularity over the functions a and b is fulfilled thanks to Proposition 2.4.

4.1 Internal measurements

Proof of Theorem 1.3, case (I). From step 1 of the proof of Proposition 3.1, let us define $w := e^{-s\alpha} v$ satisfying $L_1 w = w_t + w_{xxx} + 3s^2(\alpha_x)^2 w_x$ and $w(0,t) = w(L,t) = 0$, $w(\cdot, \pm T) = 0$ in $(0, L)$. Thus, if we multiply the expression $L_1 w$ by w and integrate it over $(0, L) \times (-T, 0)$, we have

$$\begin{aligned} J := \int_{-T}^{0}\int_0^L w L_1 w \, dx\, dt &= \int_{-T}^{0}\int_0^L w(w_t + w_{xxx} + 3s^2(\alpha_x)^2 w_x)\, dx\, dt \\ &= \frac{1}{2}\int_0^L |w(x,0)|^2\, dx - \Bigg(\underbrace{\int_{-T}^{0}\int_0^L w_x w_{xx}\, dx\, dt + 3s^2 \int_{-T}^{0}\int_0^L \phi_x \phi_{xx} \xi^2 |w|^2\, dx\, dt}_{H}\Bigg). \end{aligned}$$

Using $\phi \in C^4([0, L])$ and the Cauchy–Schwartz inequality, we have

$$|H| \le C \int_{-T}^{T}\int_0^L (s^2\xi^2|w|^2 + |w_x|^2 + |w_{xx}|^2)\, dx\, dt,$$

where C is a positive constant independent of F and $w(x, 0)$. Estimating the term sJ by Carleman inequality (3.6), by choosing s_0 large enough, the term $|H|$ can be absorbed by the left-hand side of (3.6). Therefore, the associated term to $|w(x, 0)|^2$ can be estimated by

$$\begin{aligned} s\int_0^L |w(x,0)|^2\, dx &\le 2s^2 \int_{-T}^{T}\int_0^L |w|^2\, dx\, dt + 2\int_{-T}^{0}\int_0^L |L_1 w|^2\, dx\, dt + 2s|H| \\ &\le C\Bigg(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\alpha}\, dx\, dt + \iint_{\omega\times(-T,T)} (s^5\xi^5|w|^2 + s\xi|w_{xx}|^2)\, dx\, dt\Bigg), \end{aligned}$$

where C is a positive constant independent of F.

Using the fact that α is even in time and $w = e^{-s\alpha}v = e^{-s\alpha}z^{ee}$, there exists a positive constant C independent of F such that

$$
\begin{aligned}
s\int_0^L |G(x,0)|^2 e^{-2s\alpha(x,0)}\,dx
&\le C\Big(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\alpha}\,dx\,dt + \iint_{\omega\times(-T,T)} s^5\xi^5|z^{ee}|^2 e^{-2s\alpha(x,t)}\,dx\,dt + \iint_{\omega\times(-T,T)} (s^3\xi^3|z^{ee}_x|^2 + s\xi|z^{ee}_{xx}|^2)e^{-2s\alpha}\,dx\,dt\Big)\\
&\le C\Big(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\alpha}\,dx\,dt + \int_0^T\int_\omega s^5\xi^5|z(x,t)|^2 e^{-2s\alpha(x,t)}\,dx\,dt\\
&\qquad + \int_0^T\int_\omega s^5\xi^5|z(L-x,t)|^2 e^{-2s\alpha(L-x,t)}\,dx\,dt\\
&\qquad + \int_0^T\int_\omega s^3\xi^3\big(|z_x(x,t)|^2 e^{-2s\alpha(x,t)} + |z_x(L-x,t)|^2 e^{-2s\alpha(L-x,t)}\big)\,dx\,dt\\
&\qquad + \int_0^T\int_\omega s\xi\big(|z_{xx}(x,t)|^2 e^{-2s\alpha(x,t)} + |z_{xx}(L-x,t)|^2 e^{-2s\alpha(L-x,t)}\big)\,dx\,dt\Big).
\end{aligned}
\tag{4.4}
$$

On the other hand, using the definition of $\widetilde{G^{ae}}$ and hypothesis (H2), the fact that α, ξ are even in time, and $\tilde{y}_t, y_{xt} \in L^\infty((0,L)\times(0,T))$ (see hypothesis (H1)), there exists a positive constant $C = C(L,T,M_1,M_2)$ such that

$$
\begin{aligned}
\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\alpha}\,dx\,dt &\le \int_0^T\int_0^L |G_t(x,t) - \tilde{y}_t u_x - y_{xt}u|^2 (e^{-2s\alpha(x,t)} + e^{-2s\alpha(L-x,t)})\,dx\,dt\\
&\le C\int_0^T\int_0^L |G_t(x,t)|^2 (e^{-2s\alpha(x,0)} + e^{-2s\alpha(L-x,0)})\,dx\,dt\\
&\qquad + C\int_0^T\int_0^L (|u|^2 + |u_x|^2)(e^{-2s\alpha(x,t)} + e^{-2s\alpha(L-x,t)})\,dx\,dt\\
&\le C\int_0^L |f_1(x) - f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + C\int_{-T}^{T}\int_0^L (|u^{ee}|^2 + |u^{ee}_x|^2)e^{-2s\alpha(x,t)}\,dx\,dt.
\end{aligned}
\tag{4.5}
$$

Thus, if $I_{\omega\times(0,T)}(z)$ denotes the local terms in the right-hand side of (4.4), then, from (4.4) and (4.5), we get

$$
\begin{aligned}
s\int_0^L |G(x,0)|^2 e^{-2s\alpha(x,0)}\,dx &\le C\int_0^L |f_1(x) - f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + CI_{\omega\times(0,T)}(z)\\
&\qquad + C\int_{-T}^{T}\int_0^L (|u^{ee}|^2 + |u^{ee}_x|^2)e^{-2s\alpha(x,t)}\,dx\,dt.
\end{aligned}
\tag{4.6}
$$

Finally, in order to obtain the first main result of this paper, we need to estimate the last term on the right-hand side of (4.6). To do this, we apply the Carleman inequality given in Proposition 3.1 to the equation satisfied by u^{ee}, which is the extension of (4.1) to negative times with $a(x,t) = \tilde{y}^{ee}$ and $b(x,t) = y^{ee}_x$. Thus we have

$$
\begin{aligned}
\int_{-T}^{T}\int_0^L (|u^{ee}|^2 + |u^{ee}_x|^2)e^{-2s\alpha(x,t)}\,dx\,dt &\le C\Big(\int_{-T}^{T}\int_0^L |G^{ae}|^2 e^{-2s\alpha}\,dx\,dt + \iint_{\omega\times(-T,T)} (s\xi)^5|u^{ee}|^2 e^{-2s\alpha}\,dx\,dt\\
&\qquad + \iint_{\omega\times(-T,T)} (s^3\xi^3|u^{ee}_x|^2 + s\xi|u^{ee}_{xx}|^2)e^{-2s\alpha}\,dx\,dt\Big)\\
&\le C\Big(\int_0^L |f_1(x) - f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + I_{\omega\times(0,T)}(u)\Big).
\end{aligned}
\tag{4.7}
$$

Putting together (4.6) and (4.7), we obtain

$$s\int_0^L |G(x,0)|^2 e^{-2s\alpha(x,0)}\,dx \le C\int_0^L |f_1(x)-f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + CI_{\omega\times(0,T)}(u) + +CI_{\omega\times(0,T)}(z). \tag{4.8}$$

The first result of Theorem 1.3 follows from (4.8) and assumption (H3). □

4.2 Boundary measurements

Proof of Theorem 1.3, case (II). We apply the above scheme, although using the weight functions ψ, ξ, β defined in (3.15)–(3.18) as well as the Carleman inequality with boundary terms, Proposition 3.2. Thus there exists a positive constant C independent of F such that

$$s\int_0^L |w(x,0)|^2\,dx \le C\bigg(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\beta}\,dx\,dt + \int_{-T}^{T}[s^3\xi^3|w_x(0,t)|^2 + s\xi|w_{xx}(0,t)|^2]\,dt\bigg),$$

where $w := e^{-s\beta}v$ satisfies $L_1 w = w_t + w_{xxx} + 3s^2(\beta_x)^2 w_x$ as well as $w(0,t) = w(L,t) = 0$, $w(\cdot,\pm T) = 0$ in $(0,L)$, and $v := z^e$ satisfies system (4.3).

Replacing $w := e^{-s\beta}v = e^{-s\beta}z^e$ and using the boundary conditions of z given in (4.2), we can deduce

$$\begin{aligned} s\int_0^L |G(x,0)|^2 e^{-2s\beta(x,0)}\,dx &\le C\bigg(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\beta}\,dx\,dt + \int_{-T}^{T}[s^3\xi^3|z_x^{ee}(0,t)|^2 + s\xi|z_{xx}^{ee}(0,t)|^2]e^{-2s\beta(0,t)}\,dt\bigg)\\ &\le C\bigg(\int_{-T}^{T}\int_0^L |\widetilde{G^{ae}}|^2 e^{-2s\beta}\,dx\,dt + I_{\mathrm{bo}}(z)\bigg), \end{aligned}$$

where

$$I_{\mathrm{bo}}(z) := \int_0^T [s^3\xi^3|z_x(L,t)|^2 + s\xi(|z_{xx}(0,t)|^2 + |z_{xx}(L,t)|^2)]e^{-2s\beta(0,t)}\,dt.$$

Note that estimate (4.5) associated to the source term $|\widetilde{G^{ae}}|^2$ is independent to the type of observation, and therefore we can use it. Hence there exists a positive C such that

$$\begin{aligned} s\int_0^L |G(x,0)|^2 e^{-2s\beta(x,0)}\,dx &\le C\int_0^L |f_1(x)-f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + CI_{\mathrm{bo}}(z) + C\int_{-T}^{T}\int_0^L (|u^e|^2 + |u_x^e|^2)e^{-2s\beta(x,t)}\,dx\,dt\\ &\le C\int_0^L |f_1(x)-f_2(x)|^2 e^{-2s\alpha(x,0)}\,dx + CI_{\mathrm{bo}}(u) + CI_{\mathrm{bo}}(z). \end{aligned}$$

Observe that the last estimate was obtained using the Carleman estimate with boundary term (3.19), Proposition 3.2, to the equation satisfied by u^{ee}, which is the extension of (4.1) to negative times with $a(x,t) = \tilde{y}^e$ and $b(x,t) = y_x^e$.

Finally, (H2) allows to finish the proof of the second case of Theorem 1.3. □

4.3 Measurements at a fixed time

The Lipschitz stability result given in Theorem 1.6 is proved in this subsection via Carleman estimates and employing the method developed in [8, 12] for the Navier–Stokes and Boussinesq systems, respectively. It is worthwhile to highlight that Proposition 3.1 and Proposition 3.2 hold true over $[0,L]\times(0,T)$ by replacing the weight functions (3.5) and (3.18) by $\alpha(x,t)=\phi(x)\zeta(t)$ and $\beta(x,t)=\psi(x)\zeta(t)$, respectively, where

$$\zeta(t)=\frac{1}{\ell(t)} \qquad \text{and} \qquad \ell(t)=\begin{cases} t, & 0\le t\le \frac{T}{4},\\ T-t, & \frac{3T}{4}\le t\le T,\\ \ell(t_0)>\ell(t)>0, & t\in(0,t_0)\cup(t_0,T). \end{cases} \tag{4.9}$$

Henceforth, let us denote

$$\begin{aligned} \mathcal{L}_{\mathrm{local},t_0}(u) &= \|u\|_{H^2(0,T;H^2(\omega))}+\|u(\cdot,t_0)\|_{H^3(0,L)},\\ \mathcal{L}_{\mathrm{bo},t_0}(u) &= \|u_x(0,\cdot)\|_{H^2(0,T)}+\|u_{xx}(0,\cdot)\|_{H^2(0,T)}+\|u(\cdot,t_0)\|_{H^3(0,L)}. \end{aligned}$$

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. (i) From systems (4.1) and (4.2) defined in step 1, let $w:=z_t=u_{tt}$ satisfy

$$\begin{cases} w_t+w_{xxx}-w_{xx}+\tilde{y}w_x+y_xw=G^w(x,t) & \text{in } (0,L)\times(0,T),\\ w(0,t)=w(L,t)=0 & \text{in } (0,T),\\ w_x(L,t)=w_x(0,t) & \text{on } (0,T),\\ w(\cdot,0)=w_0(\cdot) & \text{in } (0,L). \end{cases}$$

where

$$G^w(x,t):=G_{tt}(x,t)-\tilde{y}_{tt}u_x-2\tilde{y}_tu_{xt}-y_{xtt}u-2y_{xt}u_t$$

and w_0 is an appropriate initial datum.

Thus, applying the Carleman inequality with local terms (see Proposition 3.1) to u,z,w with α defined by (4.9), we can deduce

$$\begin{aligned} I(u)+I(z)+I(w) &\le C\bigg(\int_0^T\int_0^L(|G|^2+|\widetilde{G}|^2+|G^w|^2)e^{-2s\alpha}\,dx\,dt+\mathcal{L}^2_{\mathrm{local},t_0}(u)\bigg)\\ &\le C\bigg(\int_0^T\int_0^L(|G|^2+|G_t|^2+|G_{tt}|^2)e^{-2s\alpha}\,dx\,dt+\int_0^T\int_0^L|\tilde{y}_tu_x+y_{xt}u|^2e^{-2s\alpha}\,dx\,dt\\ &\qquad+\int_0^T\int_0^L|\tilde{y}_{tt}u_x+2\tilde{y}_tz_x+y_{xtt}u+2y_{xt}z|^2e^{-2s\alpha}\,dx\,dt+\mathcal{L}^2_{\mathrm{local},t_0}(u)\bigg), \end{aligned}$$

where $I(\cdot)$ is the first term in the left-hand side of Carleman inequality (3.6). Proposition 2.6 and hypothesis (H1) allow us to absorb the second and third terms in the right-hand side by the left-hand side in the above inequality. It implies the existence of a positive constant C depending on M_1 such that, for all $s\ge C$,

$$I(u)+I(z)+I(w)\le C\bigg(\int_0^T\int_0^L(|G|^2+|G_t|^2+|G_{tt}|^2)e^{-2s\alpha}\,dx\,dt+\mathcal{L}^2_{\mathrm{local},t_0}(u)\bigg). \tag{4.10}$$

On the other hand, since $\zeta(t)e^{-2s\alpha(x,t)}$ goes to zero when t tends to zero, for all $x\in[0,L]$, we have

$$\begin{aligned} \frac{s^4}{C}\int_0^L|u_t(x,t_0)|^2e^{-2s\alpha(x,t_0)}\,dx &\le s^4\int_0^L\alpha^2(x,t_0)|u_t(x,t_0)|^2e^{-2s\alpha(x,t_0)}\,dx\\ &= s^4\int_0^{t_0}\frac{d}{dt}\bigg(\int_0^L\alpha^2(x,t)|u_t(x,t)|^2e^{-2s\alpha(x,t)}\,dx\bigg)\,dt\\ &= s^4\int_0^L\int_0^{t_0}(2\alpha\alpha_t|u_t|^2+2\alpha^2u_tu_{tt}-2s\alpha^2\alpha_t|u_t|^2)e^{-2s\alpha(x,t)}\,dt\,dx\\ &\le C\int_0^L\int_0^T(s^4\zeta^3|u_t|^2+s^4\zeta^2|u_t||u_{tt}|+s^5\zeta^4|u_t|^2)e^{-2s\alpha(x,t)}\,dt\,dx. \end{aligned}$$

Note that the right-hand side is upper bounded by the left-hand side of (4.10). Besides, hypothesis (H3) allows to estimate the source terms in (4.10). Putting together this information, for all $s > 0$ large enough, we obtain

$$s^4 \int_0^L |u_t(x, t_0)|^2 e^{-2s\alpha(x,t_0)}\,dx \le C\Big(\int_0^T\int_0^L |f_1 - f_2|^2 e^{-2s\alpha}\,dx\,dt + \mathcal{L}^2_{\mathrm{local},t_0}(u)\Big). \tag{4.11}$$

Evaluating the main equation of (4.1) in $t = t_0$ for all $x \in (0, L)$,

$$G(x, t_0) = u_t(x, t_0) + u_{xxx}(x, t_0) - u_{xx}(x, t_0) + \tilde{y}(x, t_0)u_x(x, t_0) + y_x(x, t_0)u(x, t_0),$$

and using assumptions (H1) and (H3), there exists a positive constant C such that, for all $s > 0$ large enough,

$$s^4 \int_0^L |G(x, t_0)|^2 e^{-2s\alpha(x,t_0)}\,dx \le C\Big(\int_0^T\int_0^L |f_1 - f_2|^2 e^{-2s\alpha}\,dx\,dt + \mathcal{L}^2_{\mathrm{local},t_0}(u)\Big).$$

Since $-\alpha(x, t_0) \ge -\alpha(x, t)$ for all $x \in (0, L) \times (0, T)$ (see (4.9)), after using (H3), the previous inequality yields

$$s^4 \int_0^L |f_1 - f_2|^2 e^{-2s\alpha(x,t_0)}\,dx \le C\Big(\int_0^L |f_1 - f_2|^2 e^{-2s\alpha(x,t_0)}\,dx + \mathcal{L}^2_{\mathrm{local},t_0}(u)\Big),$$

and taking $s > 0$ large enough, the first term on the right-hand side can be absorbed by the left-hand side.

(ii) The proof of this case follows the previous structure. In order to obtain an inequality analogous to (4.10), we now apply the Carleman estimate with boundary terms (see Proposition 3.2) with β defined by (4.9). Thus, for all $s \ge C$,

$$J(u) + J(z) + J(w) \le C\Big(\int_0^T\int_0^L (|G|^2 + |G_t|^2 + |G_{tt}|^2)e^{-2s\beta}\,dx\,dt + \mathcal{L}^2_{\mathrm{bo},t_0}(u)\Big).$$

Here $J(\cdot)$ denotes the term on the left-hand side of Carleman inequality (3.19).

Note that (4.11) can be transformed to this case by considering β instead of α and $\mathcal{L}^2_{\mathrm{bo},t_0}(u)$ instead of $\mathcal{L}^2_{\mathrm{local},t_0}(u)$. It is implied that

$$s^4 \int_0^L |G(x, t_0)|^2 e^{-2s\beta(x,t_0)}\,dx \le C\Big(\int_0^L |f_1 - f_2|^2 e^{-2s\beta(x,t_0)}\,dx + \mathcal{L}^2_{\mathrm{bo},t_0}(u)\Big),$$

and again, using (H3) and taking $s > 0$ large enough, the first term on the right-hand side can be absorbed by the left-hand side.

Therefore, the proof of Theorem 1.6 is complete. □

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