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A source reconstruction algorithm for the Stokes system from incomplete velocity measurements

Galina C García[1], **Cristhian Montoya**[2] **and Axel Osses**[3]

[1] Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile
[2] Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/3 Correo 3, Santiago, Chile
[3] Centro de Modelamiento Matemático, UMI 2807/Universidad de Chile-CNRS, Santiago, Chile

E-mail: galina.garcia@usach.cl, cmontoya@dim.uchile.cl and axosses@dim.uchile.cl

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Abstract

We consider the inverse problem of determining the spatial dependence of a source of the form $f(x)\sigma(t)$ in the Stokes system defined in $\Omega \times (0, T)$, assuming that $\sigma(t)$ is known and $f(x)$ is divergence-free. The only available observation is a single internal measurement of the velocity and the acceleration, for which one of its components is missing. Under adequate hypothesis on σ we prove uniqueness of this inverse problem and we establish an explicit reconstruction formula. This formula provides the spectral coefficients f_k of the source f in terms of a family of null controls $h^{(\tau)}$ for the corresponding adjoint system indexed by $\tau \in (0, T]$.

Keywords: inverse source problem, null controllability, stokes system

(Some figures may appear in colour only in the online journal)

1. Introduction

Due to its importance, inverse problems for partial differential equations with incomplete or partial data have been studied intensively. Most of the time, incomplete or partial data refers to data localized in time or space, in an internal domain or on a subset of the boundary. See for instance [4, 19] for inverse problems with partial Cauchy boundary data or see for instance [20–22] for single measurement inverse problems from partial internal or boundary data using global Carleman inequalities.

Another important type of incomplete or partial data arises in those cases where the knowledge of the solution is incomplete: if some given function of the solution is known as is the case for the so called hybrid inverse problems (see [24] for a review) or if not all the components of the solution can be observed as is often the case in systems (partially observed systems [1], pressure estimation from velocity phase-contrast MRI [25], elastography [23], ocean acoustics where pressure and not velocity is measured and wireless communication where only some components of the electric field are observed [18]).

In this work, we tackle the problem of retrieving a source in an incompressible fluid from local and incomplete velocity data, i.e. from local velocity with missing components. To our knowledge, there are no available results in this setting, while there are some related results for complete velocity measurements that can be found in [5, 12] that we will discuss below. We will only consider internal measurements, but we expect the ideas of this work to be a first step to consider non intrusive incomplete boundary data (see [14]), where the identification of the source is made from measurements of the normal or the tangential component of the velocity (or the force) on a subset of the boundary.

Let us describe more precisely our problem. Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) with smooth boundary Γ. Let $T > 0$ and let $\omega \subset \Omega$ be an arbitrary nonempty subdomain. Given an initial data u_0, we consider the following Stokes system:

$$\begin{cases} u_t - \nu\Delta u + \nabla p = F(x,t) & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \Gamma \times (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $F(x,t) = f(x)\sigma(t)$ represents the source term of external forces causing the movement of the fluid and $\nu > 0$ is the kinematic viscosity of the fluid. We assume $f \in L^2(\Omega)^N$ and $\sigma \in L^2(0,T)$. It is well known that if $F \in L^2(0,T;L^2(\Omega)^N)$ and $u_0 \in V$ (V and H defined below, see for instance [15]), then there exists a unique solution (u,p) for the system (1.1) such that $u \in L^2(0,T;H^2(\Omega)^N \cap V) \cap H^1(0,T;H)$ and $p \in L^2(0,T;H^1(\Omega) \cap L^2_0(\Omega))$, where

$$H := \{u \in L^2(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \;\; u \cdot n = 0 \text{ on } \Gamma\},$$

$$V := \{u \in H^1_0(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega\}, \quad L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$$

and $n(x)$ is the outward unit normal vector to Ω at the point $x \in \Gamma$. Moreover, we will assume that $\sigma \in W^{1,\infty}(0,T)$, in particular $\sigma' \in L^2(0,T)$, so taking the time derivative in (1.1) we quickly obtain that $u \in H^1(0,T;H^2(\Omega)^N \cap V) \cap H^2(0,T;H)$ and $p \in H^1(0,T;H^1(\Omega) \cap L^2_0(\Omega))$ so for instance $u_t(T)$, $\Delta u(T)$ and $\nabla p(T)$ belong to $L^2(\Omega)$.

From now on, we assume that u_0 is known, and without loss of generality, we can assume that $u_0 = 0$ by subtracting the homogeneous solution of (1.1) from the measurements.

Our goal is to obtain a reconstruction formula for the inverse problem of determining the source $f(x)$ in the system (1.1) from local and incomplete velocity data. By local and incomplete velocity data we refer to the measurements of $N - 1$ components of the velocity field u and the acceleration field u_t in an arbitrary subset $\omega \subset \Omega$ during a time interval $(0,T)$.

In the case of complete internal velocity measurements (i.e. all the components of the velocity are measured), in [5] the authors proved the Lipschitz stability for the inverse source problem in the linearized Navier–Stokes equations with data $u|_{\omega\times(0,T)}$ and $u|_{\{\theta\}\times\Omega}$ where $\omega \subset \Omega$ in an arbitrary subdomain and $0 < \theta < T$. In that case, the external force is

$F = R(x,t)f(x)$, where $R(x,t)$ is a known vector-valued function which satisfies some non-degenaracy conditions and $f(x)$ is unknown. The proof of the Lipschitz stability relies in global Carleman inequalities and the Bukhgeim–Klibanov method [3]. In [12] the authors considered an external force of the form $F = f(x)\sigma(t)$, and they focus on recovering *f* from data $u|_{\omega\times(0,T)}$, $p|_{\omega\times(0,T)}$, $u|_{\{\theta\}\times\Omega}$, $p|_{\{\theta\}\times\Omega}$, where ω is an arbitrary subdomain and $0 < \theta < T$. The results in [12] are also based on global Carleman inequalities and the Bukhgeim–Klibanov method. In contrast with the aforementioned works, in this article we do not require the knowledge of the pressure nor the knowledge of the velocity at some given time $t = \theta$.

In the case of complete boundary measurements (i.e. all the components of the velocity or force are measured on the boundary), we also refer to the more recent work [14], where the author uses spectral analysis on an unsteady Stokes–Brinkman system in order to prove identification results for (f, g), where *f* is the external source and $g = \nabla \cdot u$ is a compressible source term. The identification is obtained from several spectral measurements of the stress tensor on the whole boundary. Similarly as in the previously cited works [5, 14], in our case we can not expect to retrieve more than the divergence-free part of the source *f* by measuring only the velocity u.

In this study we use a spectral approach that links null-controllability and inverse source problems. This method was first developed for hyperbolic equations in [17] and then extended to parabolic equations in [9] and provides an explicit reconstruction formula for the source $f(x)$ in terms of local measurements and null-controls. The main difference between the hyperbolic and the parabolic case is that in the first case, a single null-control is required to recover each component of the source, meanwhile, in the second case, a continuous family of null-controls is required. Notice that in [10], also using the connection between null controllability and inverse problems, the authors study the conditional logarithmic stability for inverse source problems, and including the Stokes system, for a wide class of parabolic equations with regular sources and from complete internal or boundary measurements. Nevertheless, the proofs in [10] do not provide explicit reconstruction algorithms.

Our main results, theorems 3.1 and 3.3, provide a reconstruction formula of each Fourier coefficient of *f* through $N - 1$ components of local measurements of the solution u of system (1.1). The reconstruction formula proposed in this work is essentially the same as the corresponding on for heat equation of [9] extended for the Stokes system. The novelties are that (a) the divergence-free part of the source *f* is retrieved from local velocity with missing components, without measuring the pressure; (b) instead of using the classical null-controllability results for evolution equations (see for instance [7, 8]), we have considered [6], where the authors obtain the null-controllability for the *N*-dimensional Stokes system with $N - 1$ scalar controls through Carleman inequalities. This fact allow us, by duality, to consider local measurements of the velocity with one missing component for the reconstruction and (c) numerically, in order to approximate a null-control with one vanishing component, it is necessary to penalize both the exact null final condition and the vanishing component of the controls and show the convergence of the regularized solutions (see [11, 13]).

The paper is organized as follows. In section 3 we first prove the uniqueness and reconstruction results, theorems 3.1 and 3.3. Next, in section 4 we give a method to approximate null controls with one vanishing component and prove its convergence. Finally, in section 5 we implement this method and present several numerical experiments that show the feasibility of the proposed reconstruction formula.

2. Preliminaries

Before starting with section 3, we recall some preliminary lemmas concerning the null controllability of Stokes system using null controls with one vanishing component.

The following result was proved in [6] and establishes the null controllability for the N-dimensional Stokes system with one vanishing component for the control using Carleman inequalities.

Lemma 2.1. *Given $\tau \in (0,T]$, $\omega \subset \Omega$ with nonempty interior and $\varphi_0 \in H$, there exists a control $h^{(\tau)} = h^{(\tau)}(\varphi_0) \in L^2(0,\tau;L^2(\Omega)^N)$ with $h_j^{(\tau)} \equiv 0$ for some $j \in \{1,\cdots,N\}$, such that the solution ϕ of the problem*

$$\begin{cases} -\phi_t - \nu\Delta\phi + \nabla\pi = h^{(\tau)} 1_{\omega\times[0,\tau]} & \text{in} \quad \Omega\times(0,\tau), \\ \nabla\cdot\phi = 0 & \text{in} \quad \Omega\times(0,\tau), \\ \phi = 0 & \text{on} \quad \Gamma\times(0,\tau), \\ \phi(\cdot,\tau) = \varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{2.2}$$

satisfies

$$\phi(\cdot,0) = 0 \text{ in } \Omega. \tag{2.3}$$

Moreover, there exist constants $C_0 > 0$ and $C_1 > 0$ depending only on Ω and ω such that

$$\|h^{(\tau)}\|_{L^2(0,T;L^2(\omega)^N)} \leqslant C_0 e^{C_1/\tau^9} \|\varphi_0\|_{L^2(\Omega)^N}. \tag{2.4}$$

Remark 2.2. The proof of lemma 2.1 is equivalent to the following observability inequality:

$$\|w(\tau)\|^2_{L^2(\Omega)^N} \leqslant C_0 e^{C_1/\tau^9} \sum_{i=1\, i\neq j}^{N} \int_0^\tau \int_\omega |w_i|^2 \mathrm{d}x\mathrm{d}t, \tag{2.5}$$

where (w,q) is the solution of the adjoint system

$$\begin{cases} w_t - \nu\Delta w + \nabla q = 0 & \text{in} \quad \Omega\times(0,\tau), \\ \nabla\cdot w = 0 & \text{in} \quad \Omega\times(0,\tau), \\ w = 0 & \text{on} \quad \Gamma\times(0,\tau), \\ w(\cdot,0) \text{ given} & \text{in} \quad \Omega. \end{cases} \tag{2.6}$$

Finally, we recall technical results about the Volterra equations of first and second kind that we will need later on. For more details, the interested reader can see [9, 16, 17].

Lemma 2.3. *For $0 < t < \tau < T$, $\sigma \in W^{1,\infty}(0,\tau)$ and $\eta \in L^2(0,\tau;L^2(\Omega)^N)$, there exists a unique*

$$\theta \in H^1(0,\tau;L^2(\Omega)^N)$$

satisfying for every $i \in \{1,\ldots,N\}$ the Volterra equation of the second kind

$$\begin{aligned} &\sigma(0)\partial_t\theta_i(x,t) + \textstyle\int_t^\tau (\sigma(s-t)\theta_i(x,s) + \sigma'(s-t)\partial_t\theta_i(x,s))\mathrm{d}s = \eta_i(x,t), \\ &\theta_i(x,\tau) = 0. \end{aligned} \tag{2.7}$$

Furthermore, there exists a constant $C > 0$ *depending on* $\|\sigma\|_{W^{1,\infty}(0,\tau)}$ *such that*

$$\|\theta\|_{H^1(0,\tau;L^2(\Omega)^N)} \leqslant C\|\eta\|_{L^2(0,\tau;L^2(\Omega)^N)}. \tag{2.8}$$

Lemma 2.4. *Assume* $\sigma \in W^{1,\infty}(\Omega)$. *We define the operator* $K : L^2(0,T;L^2(\Omega)) \to H^1(0,T;L^2(\Omega))$ *by*

$$(Kv)(x,t) := \int_0^t \sigma(s)v(x,t-s)\mathrm{d}s. \tag{2.9}$$

There exists a positive constant C depending only on Ω, T *and* $\|\sigma\|_{W^{1,\infty}(0,T)}$ *such that*

$$C\|Kv\|_{H^1(0,T;L^2(\Omega)^N)} \leqslant \|v\|_{L^2(Q)^N} \leqslant \|Kv\|_{H^1(0,T;L^2(\Omega)^N)}. \tag{2.10}$$

Furthermore, the adjoint operator $K^* : H^1(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *is given by*

$$(K^*\theta)(x,t) = \sigma(0)\partial_t\theta(x,t) + \int_t^T (\sigma(s-t)\theta(x,s) + \sigma'(s-t)\partial_t\theta(x,t))\mathrm{d}s. \tag{2.11}$$

3. Uniqueness and reconstruction with one missing component

We now address the uniqueness and the reconstruction of the inverse source problem for the Stokes system (1.1) following the same ideas of [9]. The main differences with respect to [9] are the following: first of all, we have an N-dimensional system, second, we need to project into the H space in order to eliminate the pressure term and finally, we observe the velocity field with one missing component using a null control result with one vanishing component.

Notice that every $\widetilde{f} \in L^2(\Omega)^N$ can be decomposed as $\widetilde{f} = \nabla q + f$, with $q \in H_0^1(\Omega)$ and $f \in H$ (see [15]). So it is clear that the sources $\widetilde{f}$ and f will produce the same velocity field in the Stokes system (1.1). Therefore, we can not retrieve the part of the source orthogonal to H only through velocity measurements. This is why we will always consider the H-projection of the source for the retrieval $P_H f$ where P_H represents the orthogonal projector from $L^2(\Omega)^N$ onto H.

Our first result is given in the following theorem (analogous to theorem 1.3 in [9]).

Theorem 3.1. *Let* $\sigma \in W^{1,\infty}(0,T)$ *with* $\sigma(T) \neq 0$. *Given* $\varphi_0 \in H$, *for each* $0 < \tau \leqslant T$, *let* $h^{(\tau)} = (h_j^{(\tau)})_{j=1}^N$ *be a null control associated to problem (2.2) extended by zero in* $(\tau, T]$ *with* $h_j^{(\tau)} \equiv 0$ *for some* $j \in \{1, \cdots, N\}$. *Let* $\theta^{(\tau)}$ *be a solution of (2.7) for* $\eta = h^{(\tau)}$ *extended by zero in* $(\tau, T]$. *Then*

$$(P_H f, \varphi_0)_{L^2(\Omega)^N} = \mathcal{L} + \mathcal{C}_1 + \mathcal{C}_2,$$

where

$$\begin{aligned}
\mathcal{L}(\varphi_0) &= -\tfrac{\nu}{\sigma(T)}(\Delta u(\cdot,T), \varphi_0)_{L^2(\Omega)^N}, \\
\mathcal{C}_1 &= -\tfrac{\sigma(0)}{\sigma(T)} \sum_{i=1\, i\neq j}^{N} (u_i, \theta_i^{(T)})_{H^1(0,T;L^2(\omega))}, \\
\mathcal{C}_2 &= -\tfrac{1}{\sigma(T)} \sum_{i=1\, i\neq j}^{N} \int_0^T \sigma'(T-s)(u_i, \theta_i^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.
\end{aligned} \tag{3.12}$$

Moreover, if $\sigma'(t) = 0$ *for* $t \in (T-\varepsilon, T]$ *for some* $\varepsilon > 0$ *or* $\sigma'(t) = \mathrm{e}^{-C/(T-t)^9}\rho(t)$ *for all* $t \in (0, T)$, $\rho \in L^\infty(0, T)$ *for large C, then we obtain the stability inequality*

$$\|P_H f\|_{L^2(\Omega)^N} \leqslant C\Big(\|\Delta u(\cdot, T)\|_{L^2(\Omega)^N} + \sum_{i=1\, i\neq j}^{N} \|u_i\|_{H^1(0,T;L^2(\omega))}\Big) \tag{3.13}$$

with $C \sim O(\mathrm{e}^{C_1/\varepsilon^9})$ *and* C_1 *is the constant appearing in (2.4).*

Remark 3.2. Notice that the reconstruction formula (3.12) involves a system of equations and one missing component of the velocity in the observatory $\omega \times (0, T)$ since we consider a family of exact controls $h^{(\tau)}$ having one vanishing component. This is the main difference with the reconstruction formula presented in [9] for scalar parabolic equations.

Proof of theorem 3.1. Using the operator K defined in lemma 2.4 it is easy to see that if (w, q) satisfies (2.6) with initial condition $w(\cdot, 0) = P_H f$ then $u_i = Kw_i$, $i = 1, \ldots, N$ and $p = Kq$ satisfy (1.1) with $F = P_H f(x)\,\sigma(t)$. Evaluating the main equation (1.1) in T, using that $u_t(T) = \sigma(0)w(T) + \int_0^T \sigma'(T-s)w(x, s)\mathrm{d}s$, after multiplying by $\varphi_0 \in H$ and integrating in space, we easily deduce that

$$\begin{aligned}\sigma(T)(P_H f, \varphi_0)_{L^2(\Omega)^N} = \sigma(0)(w(\cdot, T), \varphi_0)_{L^2(\Omega)^N} - \nu(\Delta u(\cdot, T), \varphi_0)_{L^2(\Omega)^N}\\ + \int_0^T \sigma'(T-s)(w(\cdot, s), \varphi_0)_{L^2(\Omega)^N}\mathrm{d}s\end{aligned} \tag{3.14}$$

since

$$(\nabla p(\cdot, T), \varphi_0)_{L^2(\Omega)^N} = 0.$$

Next, observe that for all $\tau \in (0, T]$, the term $(w(\cdot, \tau), \varphi_0)_{L^2(\Omega)^N}$ can be evaluated by multiplying the principal equation in (2.6) by ϕ, solution of the control system (2.2), and after using integration by parts in the domain $\Omega \times (0, \tau)$. Then, if $h^{(\tau)}$ is extended by zero for $\tau < t < T$ we have

$$(w(\cdot, \tau), \varphi_0)_{L^2(\Omega)^N} = -\sum_{i=1, i\neq j}^{N} \int_0^T \int_\omega w_i(x, t)h_i^{(\tau)}(x, t)\mathrm{d}x\mathrm{d}t. \tag{3.15}$$

On the other hand, from (2.7) and (2.11) we can consider the Volterra equations: $K^*(\theta_i^{(\tau)}) = h_i^{(\tau)}$, $\quad i \in \{1, \ldots, N\}, i \neq j$, where $\theta_i^{(\tau)}(t) = 0$ for $\tau \leqslant t \leqslant T$. Then, by solving these problems and using $u = Kw$ we obtain

$$(w(\cdot, \tau), \varphi_0)_{L^2(\Omega)^N} = -\sum_{i=1, i\neq j}^{N} (w_i, K^*\theta_i^{(\tau)})_{L^2(0,T;L^2(\omega))} = -\sum_{i=1, i\neq j}^{N} (u_i, \theta_i^{(\tau)})_{H^1(0,T;L^2(\omega))}.$$

Hence, applying the above identity in (3.14) for every $\varphi_0 \in H$, we have

$$\begin{aligned}(P_H f, \varphi_0)_{L^2(\Omega)^N} = -\tfrac{\sigma(0)}{\sigma(T)} \textstyle\sum_{i=1, i\neq j}^N (u_i, \theta_i^{(T)})_{H^1(0,T;L^2(\omega))} - \tfrac{\nu}{\sigma(T)}(\Delta u(\cdot, T), \varphi_0)_{L^2(\Omega)^N}\\ -\tfrac{1}{\sigma(T)} \textstyle\sum_{i=1, i\neq j}^N \int_0^T \sigma'(T-s)(u_i, \theta_i^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.\end{aligned} \tag{3.16}$$

The stability result (3.13) is deduced following the same proof as in [9] theorem 1.3, from (2.4) and (2.8) since

$$\|\theta^{(\tau)}\|_{H^1(0,\tau;L^2(\Omega)^N)} \leqslant C\|h^{(\tau)}\|_{L^2(0,\tau;L^2(\Omega)^N)} \leqslant Ce^{C_1/\tau^9}\|\varphi_0\|_{L^2(\Omega)^N}.$$

This completes the proof of theorem 3.1. □

As in [9], notice that the information of $\Delta u(\cdot,T)$ in Ω is not available in many applications, in fact, we will see that f can be recovered using information of $\Delta u(\cdot,T)$, so formula (3.12) is not useful. If we only have access to the measurements in the observatory $\omega\times(0,T)$, we can deduce the reconstruction formula of theorem 3.3.

Our second result is the following (analogous to theorem 1.6 in [9]).

Theorem 3.3. *Let $f\in L^2(\Omega)^N$ and let $\{(\lambda_k,\varphi_k)\}_{k\geqslant 0}$ be the eigenvalues and $(L^2)^N$-orthonormal eigenvectors of the Stokes operator in Ω with homogeneous Dirichlet boundary conditions. Given $\sigma\in W^{1,\infty}(0,T)$, $\sigma(T)\neq 0$, such that*

$$a_k := 1-\frac{\nu\lambda_k}{\sigma(T)}\int_0^T e^{-\nu\lambda_k(T-s)}\sigma(s)ds\neq 0, \tag{3.17}$$

for some $k\geqslant 0$, then we have the local reconstruction formula

$$(P_Hf)_k = a_k^{-1}(\mathcal{C}_{1k}+\mathcal{C}_{2k}), \tag{3.18}$$

where $\mathcal{C}_{1k}=\mathcal{C}_1(\varphi_k)$, $\mathcal{C}_{2k}=\mathcal{C}_2(\varphi_k)$ were defined in theorem 3.1, which only depend on the local observations of $N-1$ components of the solution of (1.1).

Proof of theorem 3.3. To prove the theorem 3.3 we introduce the eigenvalues and eigenvectors $(\lambda_k,\varphi_k)_{k\in\mathbb{N}}$ of the Stokes operator in Ω as follows:

$$\begin{array}{rcll} -\Delta\varphi_k+\nabla\pi_k = & \lambda_k\varphi_k & \text{in} & \Omega,\\ \nabla\cdot\varphi_k = & 0 & \text{in} & \Omega,\\ \varphi_k = & 0 & \text{on} & \Gamma, \end{array} \tag{3.19}$$

and we choose φ_k orthonormal in $L^2(\Omega)^N$ such that the solution u of (1.1) has the representation

$$u_i(x,t)=\sum_{k\in\mathbb{N}}\alpha_k(t)\varphi_{ik}(x),\quad \forall i=1,\ldots,N.$$

On the other hand, from (1.1) and (3.19) it is easy to check that the the coefficient $\alpha_k(t)$ satisfies the following identity:

$$\alpha_k(t)=f_k\int_0^t e^{-\nu\lambda_k(t-s)}\sigma(s)ds, \tag{3.20}$$

where $f_k=(f,\varphi_k)_{L^2(\Omega)^N}$ are the unknown coefficients of the source term f, which satisfies the divergence-free condition. Additionally, by integration by parts and using (3.19) and (3.20) we obtain

$$\int_\Omega \Delta u(x,T)\cdot\varphi_k(x)dx = -\lambda_k(u(\cdot,T),\varphi_k)_{L^2(\Omega)^N} = -\lambda_k\alpha_k(T). \tag{3.21}$$

Putting together (3.16), (3.20) and (3.21) we get

$$(P_H f, \varphi_k)_{L^2(\Omega)^N} := f_k = \begin{array}{l} -a_k^{-1}\Big(\sigma(0)\sigma(T)^{-1}\sum_{i=1,i\neq j}^N (u_i, \theta_{i,k}^{(T)})_{H^1(0,T;L^2(\omega))} \\ +\sigma(T)^{-1}\sum_{i=1,i\neq j}^N \int_0^T \sigma'(T-s)(u_i, \theta_{i,k}^{(s)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s\Big), \end{array}$$

where a_k was defined in (3.17). Thus the proof of theorem 3.3 is complete. □

Remark 3.4. In theorem 3.3, the reconstruction formula (3.18) is valid if the coefficient a_k defined by (3.17) is not zero. This is true for instance for every $k \in \mathbb{N}$ in the following particular cases of time dependency σ of the source (see [9]):

(a) $\sigma := \sigma_0$ constant.
(b) $\sigma := \sigma_1(t)$ a non-negative and increasing function.
(c) $\sigma := \sigma_2(t) = 1 + \frac{1}{2}\cos\left(\frac{4\pi t}{T-\varepsilon}\right)$ for $t < T - \varepsilon$ and $\sigma_2 = \frac{3}{2}$ for $t > T - \varepsilon$.

Notice that theorem 3.3 can be extended to the case in which a linear term $d(t)u(x,t)$ is added to the main equation in (1.1), with $d \in W^{1,\infty}(0,T)$. In fact, it is known that the observability inequality (2.5) is valid in the presence of this linear term in the controlled system (2.2) and the corresponding adjoint system (2.6). Thus, using the same scheme of the proof of theorem 3.3, it is easy to obtain the following corollary.

Corollary 3.5. *Under the hypothesis of theorem 3.3 and $d \in W^{1,\infty}(0,T)$, if*

$$a_k := 1 - \frac{\nu\lambda_k}{\sigma(T)}\int_0^T \mathrm{e}^{-\nu\lambda_k(T-s)+\int_s^T d(y)\mathrm{d}y}\sigma(s)\mathrm{d}s \neq 0,$$

for some $k \geqslant 0$, then we have the local reconstruction formula

$$P_H f_k = a_k^{-1}(\mathcal{C}_{1k} + \mathcal{C}_{2k} + \mathcal{C}_{3k}),$$

where P_H represents the orthogonal projector in $L^2(\Omega)^N$ onto H, $\mathcal{C}_{1k} = \mathcal{C}_1(\varphi_k)$, $\mathcal{C}_{2k} = \mathcal{C}_2(\varphi_k)$ were defined in theorem 3.1 and

$$\mathcal{C}_{3k} := -\frac{d(T)}{\sigma(T)}\sum_{i=1,i\neq j}^N \int_0^T \sigma(T-s)(u_i, \theta_{i,k}^{(\tau)})_{H^1(0,T;L^2(\omega))}\mathrm{d}s.$$

4. Convergence of optimal to null controls with one vanishing component

In this section we will study the approximation of the null controllability problem mentioned in lemma 2.1 through a sequence of optimal control problems, by introducing relaxation parameters $\alpha > 0$ and $\beta > 0$. Given $\tau \in (0,T]$, let us first characterize the control of minimal norm in $L^2(0,\tau;L^2(\Omega)^N)$ by an optimal system. For $\varphi_0 \in H$ and the index $j \in \{1,\ldots,N\}$ fixed, we consider the cost functional $J_{\alpha,\beta}$ defined by

$$J_{\alpha,\beta}(h) := \frac{1}{2}\sum_{i=1,i\neq j}^N \int_0^\tau \int_\omega |h_i|^2\mathrm{d}x\mathrm{d}t + \beta\int_0^\tau\int_\omega |h_j|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2\alpha}\|\phi(\cdot,0)\|^2_{L^2(\Omega)^N},$$

where α and β are arbitrary positive numbers, which are associated respectively to the exact final condition $\phi(\cdot, 0) = 0$ (with ϕ the solution of (2.2) with φ_0 for $t = \tau$) and the internal control with null jth component. Next, we consider the following optimal control problem:

$$\min_{h \in L^2(0,\tau;L^2(\omega)^N)} J_{\alpha,\beta}(h). \tag{4.22}$$

In [9], the authors proved a similar result of optimal control for scalar parabolic equations. The novelty here is the additional parameter β.

Theorem 4.1. *The following statements hold:*

(*i*) *For every $\alpha > 0$ and for every $\beta > 0$ there exists a unique solution $h = h(\alpha, \beta)$ to (4.22) where h is characterized by the following optimality system:*

$$\begin{cases} -\partial_t \phi - \nu \Delta \phi + \nabla \pi = h^{(\tau)} 1_{\omega \times [0,\tau]} & \text{in} \quad \Omega \times (0, \tau), \\ \nabla \cdot \phi = 0 & \text{in} \quad \Omega \times (0, \tau), \\ \phi = 0 & \text{on} \quad \Gamma \times (0, \tau), \\ \phi(\cdot, \tau) = \varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{4.23}$$

and

$$\begin{cases} \partial_t w - \nu \Delta w + \nabla q = 0 & \text{in} \quad \Omega \times (0, \tau), \\ \nabla \cdot w = 0 & \text{in} \quad \Omega \times (0, \tau), \\ w = 0 & \text{on} \quad \Gamma \times (0, \tau), \\ w(\cdot, 0) = \frac{1}{\alpha} \phi(\cdot, 0) & \text{in} \quad \Omega, \end{cases} \tag{4.24}$$

with

$$\begin{aligned} &h_i^{(\tau)} + w_i = 0 \quad \text{in} \quad \omega \times (0, \tau), \ \forall i = 1, \dots, N, \ i \neq j, \\ &\beta h_j^{(\tau)} + w_j = 0 \quad \text{in} \quad \omega \times (0, \tau). \end{aligned} \tag{4.25}$$

(*ii*) *When β tends to infinity and α tends to zero, we have*

$$\left. \begin{array}{l} -\frac{\nu}{\sigma(T)} (\Delta u(\cdot, T), \varphi_0)_{L^2(\Omega)^N} - \frac{\sigma(0)}{\sigma(T)} \sum_{i=1, i \neq j}^N (u_i, \theta_i^{(T)})_{H^1(0,T;L^2(\omega))} \\ -\frac{1}{\sigma(T)} \sum_{i=1, i \neq j}^N \int_0^T \sigma'(T - s)(u_i, \theta_i^{(\tau)})_{H^1(0,T;L^2(\omega))} \mathrm{d}s. \end{array} \right\} \to (f, \varphi_0)_{L^2(\Omega)^N},$$

where $\theta_i^{(\tau)}$ is the solution of $h_i^{(\tau)} = K^* \theta_i^{(\tau)}$.

Proof of theorem 4.1. The arguments are essentially based in [9, 11, 13], after considering the following differences:

(i) it follows from [13] that problem (4.22) has a unique solution $h^{(\tau)}$, which satisfies the optimality system (4.23)–(4.25).

(ii) From (4.23)–(4.25), it is easy to verify the identity:

$$\underbrace{\int_0^\tau \int_\omega \left(\sum_{i=1, i \neq j}^N |h_i^{(\tau)}|^2 + \beta |h_j^{(\tau)}|^2 \right) \mathrm{d}x\mathrm{d}t + \frac{1}{\alpha} \|\phi(\cdot, 0)\|_{L^2(\Omega)^N}^2}_{I_2} = (w(\cdot, \tau), \varphi_0)_{L^2(\Omega)^N}. \tag{4.26}$$

Applying Young's inequality on the right-hand side of (4.26) and combining this with the observability inequality (2.5) we obtain

$$I_2 \leqslant \frac{a^2}{2} C_0 \mathrm{e}^{C_1/\tau^9} \sum_{i=1, i\neq j}^{N} \int_0^\tau \int_\omega |w_i|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2a^2} \|\varphi_0\|^2_{L^2(\Omega)^N}, \quad a > 0.$$

Choosing $a^2 = C_0^{-1} \mathrm{e}^{-C_1/\tau^9}$ and using the optimal condition $w_i = -h_i$, $\forall i = 1, \ldots, N$, $i \neq j$, we can deduce that

$$\int_0^\tau \int_\omega \left(\sum_{i=1, i\neq j}^{N} |h_i^{(\tau)}|^2 + 2\beta |h_j^{(\tau)}|^2 \right) \mathrm{d}x\mathrm{d}t + \frac{2}{\alpha} \|\phi(\cdot, 0)\|^2_{L^2(\Omega)^N} \leqslant C_0 \mathrm{e}^{C_1/\tau^9} \|\varphi_0\|^2_{L^2(\Omega)^N}, \tag{4.27}$$

where C_0, C_1 are independent of α and β. Now, since $h_i^{(\tau)} 1_{\omega\times(0,\tau)}$ is uniformly bounded in $L^2(0, \tau; L^2(\Omega))$ for each $i = 1, \ldots, N$, $i \neq j$ and $\varphi_0 \in H$, it follows that the solution ϕ of system (2.2) is uniformly bounded in $C^0([0, \tau]; H)$ (see [15], theorem 1.1, page 172). Then, for each $n \in \mathbb{N}$ we denote by ϕ_n the solution of system (2.2) associated to $h_n^{(\tau)}$ and consider $\eta_i = h_{i,n}^{(\tau)}$ in (2.7). Thus, we can extract subsequences $\{h_{i,n'}^{(\tau)}\}$, $\{\phi_{n'}\}$, and $\{\theta_{i,n'}^{(\tau)}\}$, with $\alpha_{n'} \to 0$ and $\beta_{n'} \to \infty$ (recall that h depends on α and β), such that

$$h_{i,n'}^{(\tau)} \rightharpoonup h_i^{(\tau)} \quad \text{weakly in } L^2(0, \tau; L^2(\omega)), \quad \theta_{i,n'}^{(\tau)} \rightharpoonup \theta_i^{(\tau)} \quad \text{weakly in } H^1(0, \tau; L^2(\Omega)),$$

and

$$\phi_{n'} \rightharpoonup \phi \quad \text{weakly in } L^2(0, \tau; V), \quad \partial_t \phi_{n'} \rightharpoonup \partial_t \phi \quad \text{weakly in } L^2(0, \tau; V^*),$$

where V^* is the dual space of V. Therefore, using compactness argument for Banach spaces (see [15], theorem 2.1, page 184) we deduce that

$$\phi_{n'}(\cdot, 0) \to \phi(\cdot, 0) \quad \text{in } H, \quad n' \to +\infty. \tag{4.28}$$

On the other hand, from (4.27) we have

$$\beta \|h_j^{(\tau)}\|^2_{L^2(0,\tau;L^2(\omega))} \leqslant C_0 \mathrm{e}^{C_1/\tau^9} \|\varphi_0\|^2_{L^2(\Omega)^N} \quad \text{and} \quad \|\phi_{n'}(\cdot, 0)\|_{L^2(\Omega)^N} \to 0,\ n' \to \infty,$$

this implies that $h_j^{(\tau)}$ is uniformly bounded in $L^2(0, \tau; L^2(\omega))$ and thanks to (4.28), $\phi(\cdot, 0) = 0$ in Ω. Moreover, if $\beta \to +\infty$ then $h_j^{(\tau)} \to 0$ in $L^2(0, \tau; L^2(\omega))$. Finally, for fixed $\varphi_0 \in H$ we find:

$$\left\{ \begin{aligned} &-\tfrac{\nu}{\sigma(T)} (\Delta u(\cdot, T), \varphi_0)_{L^2(\Omega)^N} - \tfrac{\sigma(0)}{\sigma(T)} \textstyle\sum_{i=1,i\neq j}^{N} (u_i, \theta_{i,n'}^{(T)})_{H^1(0,T;L^2(\omega))} \\ &-\tfrac{1}{\sigma(T)} \textstyle\sum_{i=1,i\neq j}^{N} \int_0^T \sigma'(T-s) (u_i, \theta_{i,n'}^{(\tau)})_{H^1(0,T;L^2(\omega))} \mathrm{d}s \end{aligned} \right\} \to (f, \varphi_0)_{L^2(\Omega)^N},$$

which concludes the proof of theorem 4.1. □

5. Numerical source reconstruction

In this section we propose a two dimensional numerical algorithm of the reconstruction formula (3.18) established in theorem 3.3 that can be easily adapted to three or higher dimensions. The formula allows to reconstruct the H-projection of the unknown spatial dependency of the source $f(x)$ for the Stokes system (1.1) from observations of one component of the velocity in a subdomain $\omega \times (0, T)$. The objective is to test the feasibility of the formula for different choices of the known temporal dependency of the source $\sigma(t)$ (see remark 3.4).

Notice that we have to solve several null controllability problems (see (2.2)) and Volterra integral equations (2.11) in order to compute the projections of $f \in L(\Omega)^2$ on some given direction $\varphi_k \in H$. The numerical scheme to solve each Volterra equation is the same as in [9]. On the other hand, the null-controls with one vanishing component are approximated by using the two-parameter optimal controls introduced in the previous section. More precisely, we implement the following algorithm:

Remark 5.1. Taking into account (4.23), let us first introduce $(\bar{\psi}, \bar{\pi})$ and $(\hat{\psi}, \hat{\pi})$, the corresponding solutions of the following systems:

$$\begin{cases} -\partial_t \bar{\psi} - \nu \Delta \bar{\psi} + \nabla \bar{\pi} = 0 & \text{in} \quad \Omega \times (0, \tau), \\ \nabla \cdot \bar{\psi} = 0 & \text{in} \quad \Omega \times (0, \tau), \\ \bar{\psi} = 0 & \text{on} \quad \Gamma \times (0, \tau), \\ \bar{\psi}(\cdot, \tau) = \varphi_0 & \text{in} \quad \Omega, \end{cases} \tag{5.29}$$

and

$$\begin{cases} -\partial_t \hat{\psi} - \nu \Delta \hat{\psi} + \nabla \hat{\pi} = h^{(\tau)} 1_{\omega \times [0,\tau]} & \text{in} \quad \Omega \times (0, \tau), \\ \nabla \cdot \hat{\psi} = 0 & \text{in} \quad \Omega \times (0, \tau), \\ \hat{\psi} = 0 & \text{on} \quad \Gamma \times (0, \tau), \\ \hat{\psi}(\cdot, \tau) = 0 & \text{in} \quad \Omega. \end{cases} \tag{5.30}$$

Now, let us consider the linear operators $L : H \to L^2(0, \tau; L^2(\omega)^2)$ and $L^* : L^2(0, \tau; L^2(\omega)^2) \to H$ defined by

$$Lw(\cdot, 0) := -w 1_{\omega \times [0,\tau]} \quad \text{and} \quad L^* h^{(\tau)} := -\hat{\psi}(\cdot, 0),$$

where w is the solution of (4.24) with initial condition $w(\cdot, 0)$ and $\hat{\psi}$ is the solution of (5.30). Furthermore, we consider the linear operator $\Lambda = L^* L : H \to H$ defined by

$$\Lambda w(\cdot, 0) := -I_\beta^{(j)} \hat{\psi}(\cdot, 0),$$

for either $j = 1$ or $j = 2$, where

$$I_\beta^{(1)} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_\beta^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

Thus, the solution of the optimal control system (4.23)–(4.25) is given by the unique solution of:

$$\text{Find } w(\cdot, 0) \in H \text{ such that } (\alpha I + I_{1/\beta}^{(j)} \Lambda) w(\cdot, 0) = \bar{\psi}(\cdot, 0). \tag{5.31}$$

In the previous scheme, as we have already mentioned, the null exact final condition is penalized by α and the vanishing component of the control is penalized by the second parameter β.

The finite dimensional version for the operator Λ is based on the time-space discretization of system (4.23)–(4.25). More precisely, we consider finite differences for the time discretization and a mixed finite element formulation in space using $\mathbb{P}_2$-type elements for the velocity and $\mathbb{P}_1$-type elements for the pressure which the classical finite element spaces of piecewise polynomials (see e.g. [2, 11]).

For the sake of clarity, we list all the steps involved in the reconstruction algorithm:

- compute the matrix associated to the operator Λ: in the *j*th column of the matrix we put the solution of (4.23)–(4.25) with the *j*th basis finite element function as initial condition.
- Compute the first M eigenvalues and eigenvectors $(\lambda_k.\varphi_k)$, $k = 1, \ldots, M$, of the Stokes system (3.19).
- For each eigenvector φ_k, compute the solution of (5.29) with initial condition $\varphi_0 = \varphi_k \in H$. Next, given the parameters α, β and $\bar{\psi}(\cdot, 0)$ solve (5.31) to obtain $w(\cdot, 0)$.
- In order to obtain the optimal control $h^{(\tau)}$, solve (4.24) with initial condition $w(\cdot, 0)$, obtained from the previous step by considering (4.25), for each $\tau \in (0, T]$.
- For each control $h^{(\tau)}$, we compute the Volterra equation $K^*\theta^{(\tau)} = h^{(\tau)}$ (recall to see the discretization of (2.11) in [9]), to obtain $\theta^{(\tau)}$ for some discretized set $\tau \in (0, T]$.
- Finally, use (3.18) to find the coefficients of the source $P_H f$. This completes the application of the reconstruction formula (3.18).
- Apply, if needed, an extra optimization method (5.32). See the discussion below.

In practice, we observe that the numerical results obtained with the formula (3.18) allow us to detect with some accuracy the position of the source but not its amplitude. Therefore we implement an additional step consisting on a classical optimization algorithm that minimizes the fit between predicted and measured observations, but restricted to the frequencies associated to the significant coefficients found in the previous step. More precisely:

$$
\begin{aligned}
P_H \hat{f} &= \sum_k \hat{c}_k f_k \varphi_k \\
\hat{c} &= \operatorname{argmin}_{c \in \mathbb{R}^{M'},\ g(c) = \sum_k c_k f_k \varphi_k} \|u^m - u(g)\|^2_{H^1(0,T;L^2(\omega)^2)} + \mu \|g - P_H f\|^2_H,
\end{aligned}
\tag{5.32}
$$

where u^m are the given measurements, $\mu > 0$ is some regularization parameter and $P_H f$ is the recovered source using the reconstruction formula (3.18) for $0 \leqslant k \leqslant M$. In fact, we only optimize, starting from 1, the factors c_k for which $|f_k| > \varepsilon$, for some given threshold $\varepsilon > 0$ (which are renumbered c_k, $k = 1, \ldots, M'$, with $M' \leqslant M$).

For the numerical experiments we use the following data: we fix $\Omega = (0, 1) \times (0, 1)$, $T = 1$, the mesh size for the finite difference method is $h = 0.05$ and the time step size is $\Delta t = 5 \times 10^{-3}$. We choose $M = 38$ as the maximum number of eigenfrequencies for which the minimum wavelength reached is around $0.25 = 5h$. This number corresponds also to the maximum frequency for which the null controllability is effective, that is to say, the numerical control drives the solution to zero faster than in the situation of natural decay without control. The observation set ω is either $(0, 1) \times (0.3, 0.7)$ (centered) or $(0, 1) \times [(0, 0.1) \cup (0.9, 1)]$ (near the boundary). The diffusion parameter is $\nu = 5 \times 10^{-2}$ and the regularization parameters are $\alpha = 5 \times 10^{-3}$ and $\beta = 15$.

We consider a divergence-free unknown source of the form $f = (-\partial_2 g, \partial_1 g)$ (so $P_H f = f$), where g is a Gaussian function with small deviation (dipole type source) localized either at the top of the domain (single dipole) or at the top and bottom of the domain (double dipole) where

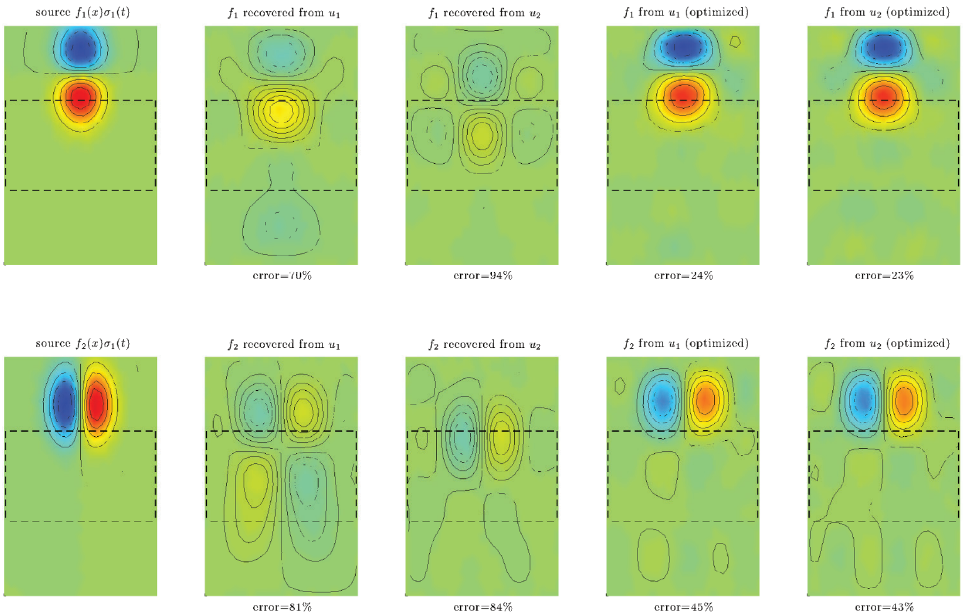

Figure 1. Source reconstruction of a single located dipole source from measurements in a centered observatory (dashed line). The color scale is common for all the figures from −3 (blue) to 3 (red).

we have added an additional source with smaller deviation near the limit that can be still well represented with the selected number of eigenfunctions.

In figures 1–3 we show different examples of source reconstruction using the methodology proposed in the present study. In all the cases the source is of the form $f(x)\sigma(t)$, where $f = (f^1, f^2)$ is an unknown divergence-free function, and σ is the known time dependency $\sigma = \sigma_1(t)$ for an increasing type or $\sigma = \sigma_2(t)$ for an oscillating type (see remark 3.4). For the reconstructions, we only use measurements of a single component of the velocity, that is either u_1 or u_2 restricted in space to a local observatory (marked with a dashed line in the corresponding figures) and with 5% additive random noise. We first estimate the components f^1 and f^2 of the source using the null control reconstruction formula presented in (3.18) (second and third columns in all the figures) and then use that first estimate as a first guess for the optimization algorithm (5.32) (last two columns), where only the previously non negligible coefficients are further adjusted. The relative error of each reconstruction is computed as the L^2-norm of the difference between the original and the reconstructed source divided by the L^2-norm of the original source, in percentage. Additionally, the source spectrum corresponding to the examples of figures 1 and 3 are shown in figures 4 and 5.

In all the examples, the reconstruction before the optimization step is better by using measurements of the first horizontal component of the velocity u_1 than by using the second vertical component u_2. In the first case the location of the source is better established than in the second. In both cases the amplitude is underestimated, which is a consequence with the fact that the considered observatories are wider in the horizontal direction and the underestimation of the amplitude is expected in such type of logarithmical ill-posed problems. In all the cases, the presence of higher random noise until 10% shows similar reconstructions. This is due to the fact that random noise is introduced as random perturbations in space at each node

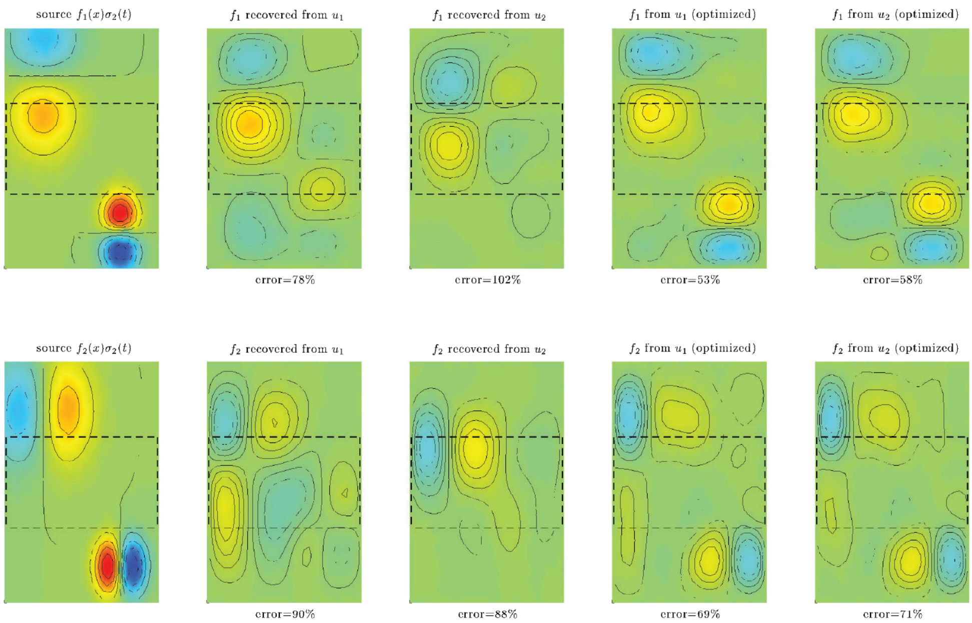

Figure 2. Source reconstruction of a double located dipole source from measurements in a centered observatory (dashed line).

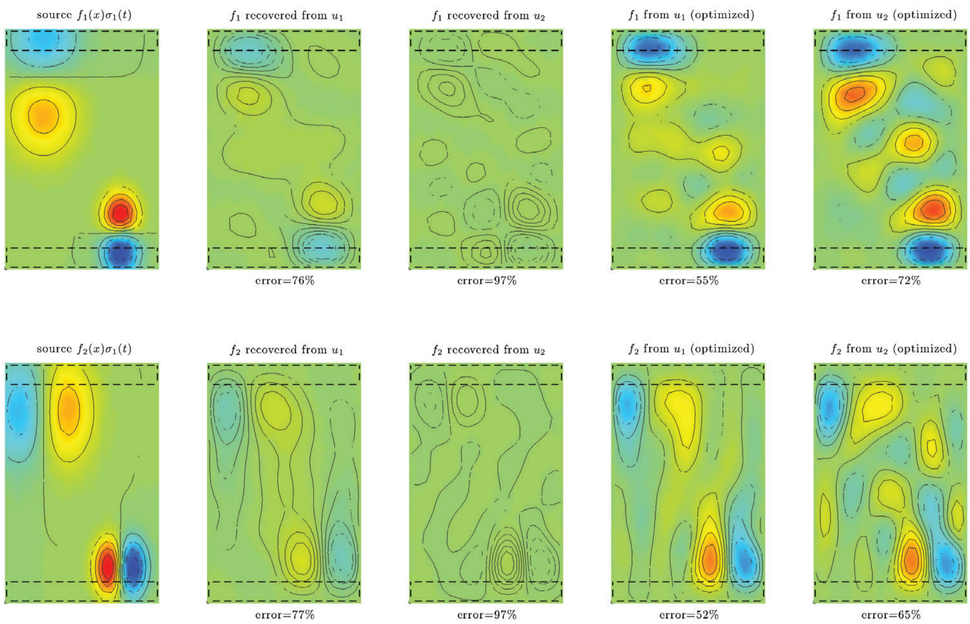

Figure 3. Source reconstruction of a double located dipole source from measurements in a near boundary observatory (dashed line).

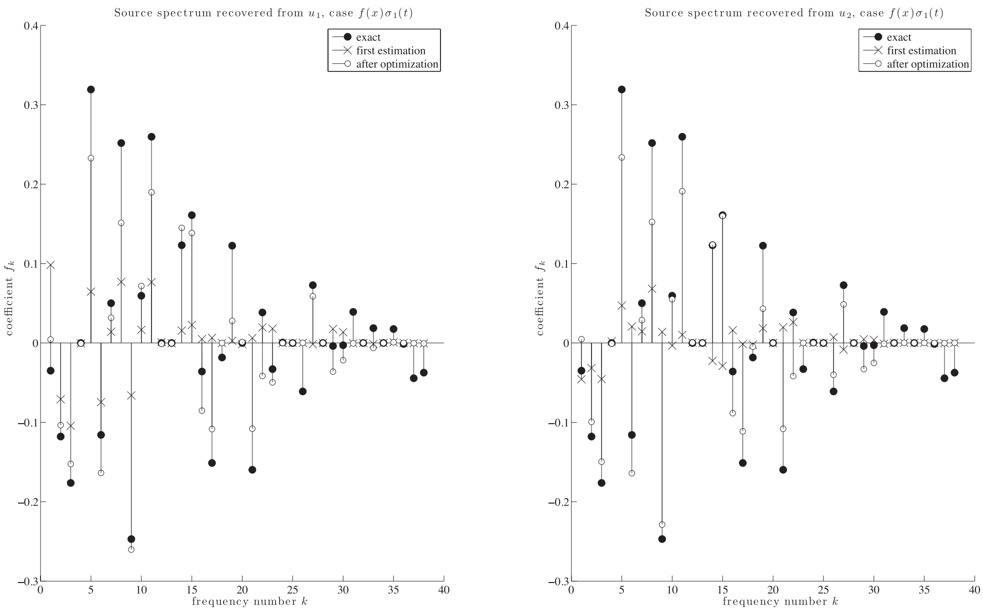

Figure 4. Spectrum of the reconstructed source for the example of figure 1 from the first (left) or second (right) component of the velocity. Exact (dark circles), obtained from the reconstruction formula (3.18) (cross) and after optimization step (5.32) (white circles).

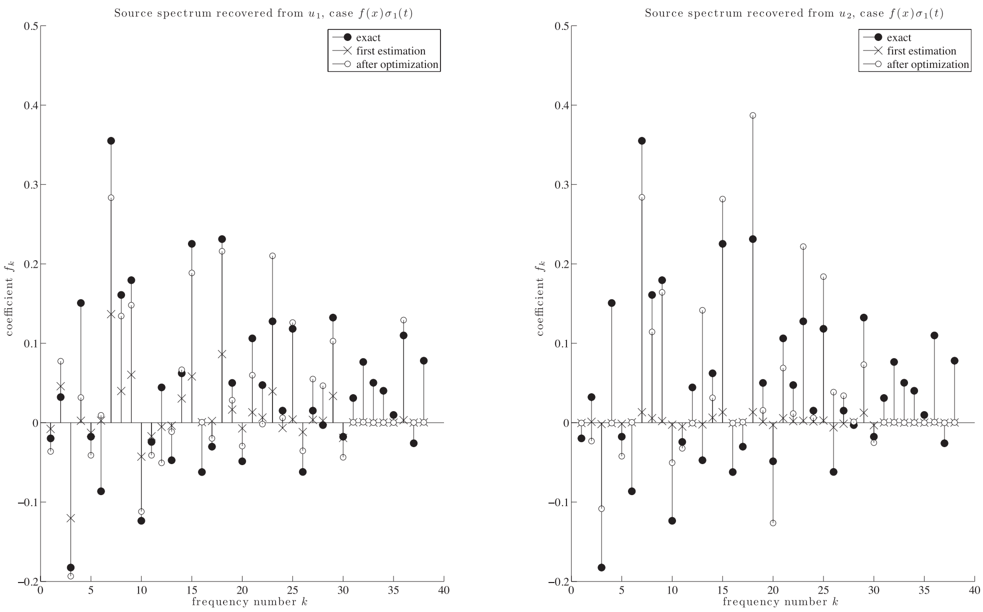

Figure 5. Spectrum of the reconstructed source for the example of figure 3 from the first (left) or second (right) component of the velocity.

of the mesh (for the measured velocity and acceleration), which is in fact a high frequency noise associated to the mesh size that is filtered as we work with the H-projection in the first M eigenfrequencies, i.e. the mesh wavelength $2h$ is smaller than minimum wavelength presented in the eigenfunctions which is around $5h$ as mentioned before.

In all the cases, the optimisation process allow us to improve the implitudes of the corresponding unknown sources in the case of u_1 or u_2 measurements and also the location of the source in the case of u_2 measurements. Moreover, we can observe in each case a significative decrease the relative error. This is clear in figure 4 where the source spectrum of the example of figure 1 is shown. The optimization step allows to correct (generally by amplifying) the amplitudes of the first estimation of the spectral coefficients from the null-control recovery formula. This effect is more evident in the example of figure 2 in which the reconstruction formula does not allow to clearly distinguish the source of the bottom before the optimization step, but the high frequency coefficients associated with this second source are over the ε-threshold and this allows to amplify them in the optimization step as is depicted in figure 5 and therefore enhance the reconstruction of this second source after optimization. Nevertheless, in some cases as is shown in figure 3, the optimization step could also amplify some spurious sources far away from the observatory.

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